THE r-STABILITY OF HYPERSURFACES WITH ZERO GAUSS-KRONECKER CURVATURE

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ABSTRACT. In this paper we give sufficient conditions for a bounded domain in a r-minimal hypersuface of the Euclidean space to be r-stable. The Gauss-Kronecker curvature of this hypersurfaces may be zero on a set of capacity zero.

1. INTRODUCTION

Let M be an oriented hypersurface of the $(n + 1)$-dimensional Euclidean space and let $g : M \to S^n$ denote its Gauss map. The shape operator of M is the self-adjoint map given by $B := -dg$, that is, for each $p \in M$,

$$B_p : T_pM \to T_pM, \ B_p(X) = -dg_p(X).$$

The eigenvalues of B_p are called the *principal curvatures of M at p*. We denote them by $k_1(p), \ldots, k_n(p)$ and we define the *r-mean curvature of M* as the normalized r-elementary symmetric function of the principal curvatures of M, namely,

$$H_0 = 1, \ H_r = \binom{n}{r}^{-1} S_r, \ r = 1, \ldots, n,$$

where $S_r = \displaystyle\sum_{i_1 < \ldots < i_r} k_{i_1} \ldots k_{i_r}$. Notice that H_1, H_2, H_n are the mean, scalar and Gauss-Kronecker curvatures of M, respectively.

We say that M is *r-minimal* when $H_{r+1} = 0$. It is well known that the r-minimal hypersurfaces of the Euclidean space are critical points of the r-area functional $A_r = \int_M S_r dM$ for compactly supported variations of M.

In order to state our results we need more notations. Let P_r be the *Newton transformations of B*, which can be defined inductively by

$$P_0 = I, \ P_r = S_r I - B P_{r-1}, \quad r = 1, 2, \ldots n.$$

Let $C_0^\infty(M)$ denote the set of smooth functions with compact support on M. Using the Newton transformations we define *the linear operator*

$$L_r(f) = \mathrm{div}(P_r \nabla f), \quad \text{for } f \in C_0^\infty(M).$$

We denote by T_r the *Jacobi operator*,

$$T_r f = L_r f - (r + 2) S_{r+2} f,$$

and by $I_r(f_1, f_2) = -\int_M f_1 T_r(f_2) dM$ the associated bilinear symmetric form.

Let D be a regular domain on M, that is, D is bounded and has piecewise smooth boundary. Following [1], we say that D is r-*stable* if $I_r(f, f) \geq 0$ for all $f \in C_0^\infty(D)$ or if $I_r(f, f) \leq 0$ for all $f \in C_0^\infty(D)$. Otherwise we say that D is r-*unstable*.

In the study of r-stability we need to suppose that L_r is elliptic. This is equivalent to P_r being positive definite or negative definite everywhere. On the other hand, by a Theorem of Hounie-Leite in [7], it is known that, when $H_{r+1} = 0$, then L_r is elliptic if and only if $\operatorname{rank}(B) > r$. In the following, without loss of generality, we will fix $P_r > 0$. Also, the eigenvalues of the operator $\sqrt{P_r}B$ appear naturally and we will denote them by $\theta_1(r), \theta_2(r), ..., \theta_n(r)$.

In [1] Alencar, do Carmo and Elbert gave sufficient conditions for a regular domain on a r-minimal hypersurface of the Euclidean space to be r-stable. Their general result assumes that the quotient $\frac{|H_n|}{\|\sqrt{P_r}B\|^2}$ is constant. In this case the hypersurface is said to be r-*special*.

Theorem A (Theorem 1.3 of [1])**.** Let $x : M^n \to \mathbb{R}^{n+1}$ be an oriented r-special hypersurface with $H_{r+1} = 0$ and $H_n \neq 0$ everywhere. Let $D \subset M$ be a regular domain such that the area of $g(D) \subset S^n$ is smaller than the area of a spherical cap whose first eigenvalue for the spherical Laplacian is

$$\tau = \max_{i, D} \left(\frac{\sum_j \theta_j^2(r)}{\theta_i^2(r)} \right).$$

Then D is r-stable.

This Theorem is a generalization of a classical result to minimal surfaces of $\mathbb{R}^3$ due to Barbosa and do Carmo in [3]. We point out that the hypothesis of M to be r-special occurs naturally when $r = n - 2$. In fact, a computation shows that $\frac{|S_n|}{\|\sqrt{P_{n-2}}B\|^2} = \frac{1}{n}$.

In this paper we are interested to improve the condition on the Gauss-Kronecker curvature considering points on M for which $H_n = 0$. A simple example, like a cylinder over a plane curve, shows that H_n cannot be identically zero. In [1], remark 4.2, was conjectured that Theorem A holds if the set of zeros of H_n is contained in a submanifold of codimension $d \geq 2$. Here we answer this conjecture affirmatively as a consequence of a more general result. We will consider hypersurfaces with $H_n = 0$ on a subset of *capacity* zero (see below the definition of capacity). It is known (see [8] §2, p. 35) that submanifolds of codimension $d \geq 2$ have capacity zero. In section 2 we will give definitions and develop some facts about capacity. To state our result we will denote by A the set $A = \{p \in M : H_n(p) = 0\}$ and by $\lambda_1(D)$ the *first eigenvalue* for the Laplacian on D. We also point out that, since symmetrization of domains in the sphere does not increase eigenvalues, the hypothesis on the first eigenvalue of the spherical image of D in Theorem A implies that $\lambda_1(g(D)) \geq \tau$. For simplicity we will use this condition on our result below.

Theorem 1.1. *Let M be an oriented r-minimal hypersurface of $\mathbb{R}^{n+1}$, which is r-special on $M \setminus A$. Let $D \subset M$ be a regular domain such that $\lambda_1(g(D)) \geq \tau$. Then, if $\operatorname{Cap}(D \cap A) = 0$, D is r-stable.*

The idea of the proof is to use a relation between the eigenvalues of domains from which we remove a subset and the capacity of the removed subset. Actually, we need just a comparison between the first eigenvalues of P_r on D and on $D \setminus A$.

This relation is well known for the Laplacian and we can find in [8], for domains of the Euclidean space, and in [4], for domains of a closed Riemannian manifold. Here we obtain such results for an elliptic operator L in divergence form on a bounded domain D of a n-dimensional Riemannian manifold M. Given $A \subset D$, let $\lambda_k(D)$ and $\lambda_k(D \setminus A)$ denote the k-th eigenvalue of the Dirichlet problem of L on D and on $D \setminus A$, respectively. We have the following result that will be proved in the next section

Theorem 1.2. *For an elliptic operator L in divergence form there exist positive constants ε_k and C_k, such that if $\operatorname{Cap} A \leq \varepsilon_k$, then*

$$\lambda_k(D) \leq \lambda_k(D \setminus A) \leq \lambda_k(D) + C_k \operatorname{Cap} A^{\frac{1}{2}}.$$

In particular, $\lambda_k(D) = \lambda_k(D \setminus A)$ if $\operatorname{Cap} A = 0$.

2. The Spectrum of Domains and the Capacity

In this section we introduce the notion of capacity and recall some spectral properties of domains in Riemannian manifolds. In this section we also prove Theorem 1.2.

Let $(M^n, \langle\,,\,\rangle)$ be a smooth Riemannian manifold and $D \subset M$ a bounded domain. As usual, we define $H_0^1(D)$ as the closure of $C_0^\infty(D)$ with respect to the norm H^1:

$$|u|_{H^1}^2 := \int_D u^2 dM + \int_D |\nabla u|^2 dM, \text{ for } u \in C_0^\infty(D)$$

where ∇ is the gradient, $|.|$ is the norm of a vector, and dM is the volume element with respect to the metric $\langle\,,\,\rangle$. Now, given $A \subset D$, we set

$$\mathfrak{H}(D; A) = \{u \in H_0^1(D) : \exists U \subset D \text{ open, } A \subset U \text{ and } u = 1 \text{ in } U\}$$

and $H(D; A)$ as the *closure* of $\mathfrak{H}(D; A)$ with respect to H^1. With this notation we define the *Capacity of A* as

$$\operatorname{Cap} A = \inf \left\{ \int_D |\nabla u|^2 dM : u \in H(D; A) \right\}.$$

Below we have some well known consequences of this definition (see, e.g. [5], §4.7).

Proposition 2.1. For any $A \subset D$ we have,
i) $\operatorname{Cap} A = \inf\{\operatorname{Cap} U : U \text{ open, } A \subset U\}$;
ii) If $A_1 \subset \cdots \subset A_k \subset \ldots$ are compact subsets of D, then

$$\lim_{k \to \infty} \operatorname{Cap} A_k = \operatorname{Cap}\left(\cap_k A_k\right).$$

We will use the notation $V^\varepsilon(A)$ to mean the *tubular neighborhood of A* of radius ε, that is, $V^\varepsilon(A) = \{x \in M : \operatorname{dist}(x, A) < \varepsilon\}$, where $\operatorname{dist}(\,,\,)$ stands for the distance function on M. Using this notation we have the following consequence of the above proposition:

Corollary 2.2. If A is compact, then

$$\lim_{\varepsilon \to \infty} \operatorname{Cap} V^\varepsilon(A) = \operatorname{Cap} A.$$

In the case that A is a submanifold of codimension $d \geq 2$ we can say more:

Proposition 2.3. If A is an embedded submanifold of codimension $d \geq 2$ then there exists a family of open sets $V_j \supset A$ such that $\cap_j V_j = A$ and

$$\text{Cap } V_j \to 0 \text{ as } j \to \infty.$$

In particular, $\text{Cap } A = 0$.

This proposition follows from minor modifications of Theorem 3, page 154 of [5]. Now recall that, $H_0^1(D)$ is a Hilbert space with respect to

$$\langle u_1, u_2 \rangle_* = \int_D \langle \nabla u_1, \nabla u_2 \rangle dM.$$

In fact $\langle \,, \rangle_*$ and $\langle \,, \rangle_{H^1}$ yield equivalent norms, as we can see by using the *Poincaré inequality*

$$|u|_{L^2(D)} \leq C|\nabla u|_{L^2(D)}.$$

Hence $H(D; A)$ is a closed (affine) subspace of $H_0^1(D)$ with respect to $\langle \,, \rangle_*$. Let u_A be the orthogonal projection of 0 on $H(D; A)$. By definition we get $|\nabla u_A|^2_{L^2(D)} = \text{Cap } A$.

Now we start the proof of Theorem 1.2 following some ideas contained in [4]. Here we will consider nonempty boundary domains and general elliptic operators in the divergence form.

For each $x \in D$, let $P_x : T_x M \to T_x M$ be a symmetric, positive (or negative) definite operator. We define, for each $f \in C_0^\infty(D)$,

$$Lf = \text{div}(P\nabla f) + qf,$$

where $q : D \to \mathbb{R}$ is a bounded function.

We consider the unique extension of L to $H_0^1(D)$. Then, L is an elliptic operator and let us denote by $\{\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\}$ the spectrum of the Dirichlet problem of L on D, repeated according to its multiplicity.

We recall Courant's min-max principle for the eigenvalues of L on domains of M:

$$\lambda_k(D \setminus A) = \min_{E \in \mathcal{E}_k} \max_{f \in E \setminus \{0\}} \frac{\int_{D \setminus A}(\langle P\nabla f, \nabla f \rangle + qf^2)dM}{\int_{D \setminus A} f^2 dM},$$

where $\mathcal{E}_k$ is the set of k-dimensional subspaces of $H_0^1(D \setminus A)$. The quotient above is called *the Rayleigh quotient* for f.

In order to simplify some calculations below we set

$$Q(f; D) := \int_D (\langle P\nabla f, \nabla f \rangle + qf^2)dM.$$

So, the Rayleigh quotient for f in D is given by $Q(f; D)/|f|^2_{L^2(D)}$.

By using the analogous characterization for $\lambda_k(D)$ we easily obtain the first inequality of Theorem 1.2. In order to obtain the second inequality we choose $f_1, \dots, f_k$, an orthonormal basis of eigenfunctions associated to $\lambda_1(D), \dots, \lambda_k(D)$, and set F_k the space generated by $f_1, \dots, f_k$. Then $F_k \subset H_0^1(D)$ and

$$\lambda_k(D) = \max_{f \in F_k \setminus \{0\}} \frac{Q(f; D)}{|f|^2_{L^2(D)}}.$$

We now define $E_k = \{g = f(1 - u_A) : f \in F_k\}$. It is clear that $E_k \subset H_0^1(D \setminus A)$ is a finite dimensional subspace. We will see that, when A has small capacity, then

E_k has dimension equal to k. In fact, the functions $g_j = f_j(1 - u_A), j = 1, \ldots, k$ form a basis for E_k. We have that

$$\langle g_i, g_j \rangle_{L^2(D)} = \delta_{ij} - 2 \int_D f_i f_j u_A dM + \int_D f_i f_j u_A^2 dM.$$

Thus, using the Cauchy-Schwarz and Poincaré inequalities we have

$$|\langle g_i, g_j \rangle_{L^2(D)} - \delta_{ij}| \leq 2 \Big| \int_D f_i f_j u_A dM \Big| + \Big| \int_D f_i f_j u_A^2 dM \Big|$$

$$\leq 2 \Big(\int_D (f_i f_j)^2 dM \Big)^{\frac{1}{2}} \Big(\int_D u_A^2 dM \Big)^{\frac{1}{2}} + \max_D (f_i f_j) \int_D u_A^2 dM$$

$$\leq C_k \Big(\int_D u_A^2 dM \Big)^{\frac{1}{2}} + C_k' \int_D u_A^2 dM$$

$$\leq B_k (\operatorname{Cap} A^{\frac{1}{2}} + \operatorname{Cap} A),$$

where B_k is a positive constant depending only on $\operatorname{vol} D$ and $\max\limits_{i=1,\ldots,k} |f_i|_{L^\infty(D)}$. On the other hand, we may choose $\varepsilon_k > 0$ sufficiently small such that, if $\operatorname{Cap} A < \varepsilon_k$ then

$$B_k (\operatorname{Cap} A^{\frac{1}{2}} + \operatorname{Cap} A) < \min\{\langle g_i, g_j \rangle_{L^2(D)} - \delta_{ij} : i, j = 1, \ldots, k\}.$$

For such ε_k we have that $g_1, \ldots, g_k$ form an orthonormal basis and consequently E_k has dimension equal to k as we claimed. Now we look for estimates of the numerator of the Rayleigh quotient for $g = f(1 - u_A) \in E_k$, where $f \in F_k$ and $|f|_{L^2(D)} = 1$. We first observe that for any closed subset $A \subset D$.

(2.1) $$|g|_{L^2(D)}^2 = 1 - 2 \int_D f^2 u_A dM + \int_D f^2 u_A^2 dM \geq 1 - B_k' \operatorname{Cap} A^{\frac{1}{2}}.$$

Now we have,

$$Q(g; D) = \int_D \big([\langle P \nabla f, \nabla f \rangle + q f^2] - 2 \langle P \nabla f, \nabla f \rangle u_A + \langle P \nabla f, \nabla f \rangle u_A^2 \big) dM$$

$$+ \int_D \big(\langle P \nabla u_A, \nabla u_A \rangle f^2 - 2 \langle P \nabla f, \nabla u_A \rangle f(1 - u_A) - 2 f^2 u_A + f^2 u_A^2 \big) dM$$

$$\leq \lambda_k(D) + B_k'' (\operatorname{Cap} A + \operatorname{Cap} A^{1/2}),$$

where B_k'' depends only on $\operatorname{vol} D$, $\max_{i=1,\ldots,k} |f_i|_{L^\infty(D)}$, $\max_{i=1,\ldots,k} |\nabla f_i|_{L^\infty(D)}$ and $\max_D ||P||$.

Therefore, choosing $\varepsilon_k > 0$ such that $|g|_{L^2(D)}^2 > 0$ in (2.1), we have

$$\frac{Q(g; D)}{|g|_{L^2(D)}^2} \leq \lambda_k(D) + \frac{\lambda_k(D) B_k' \operatorname{Cap} A^{1/2} + B_k''(\operatorname{Cap} A + \operatorname{Cap} A^{1/2})}{1 - B_k' \operatorname{Cap} A^{1/2}}.$$

We observe that, if $\operatorname{Cap} A < \varepsilon_k$, then we can estimate the second term above

$$\frac{\lambda_k(D) B_k' \operatorname{Cap} A^{1/2} + B_k''(\operatorname{Cap} A + \operatorname{Cap} A^{1/2})}{1 - B_k' \operatorname{Cap} A^{1/2}} \leq \frac{\lambda_k(D) B_k' \operatorname{Cap} A^{1/2}}{1 - B_k' \varepsilon_k^{1/2}} +$$

$$\frac{(B_k'' \operatorname{Cap} A^{1/2} + B_k'') \operatorname{Cap} A^{1/2}}{1 - B_k' \varepsilon_k^{1/2}} \leq \frac{(\lambda_k(D) B_k' + B_k'' \varepsilon_k^{1/2} + B_k'') \operatorname{Cap} A^{1/2}}{1 - B_k' \varepsilon_k^{1/2}}$$

$$= C_k \operatorname{Cap} A^{1/2}.$$

So,

$$\frac{Q(g; D)}{|g|^2_{L^2(D)}} \leq \lambda_k(D) + C_k \operatorname{Cap} A^{1/2}.$$

This implies that

$$\lambda_k(D \setminus A) := \max_{g \in E_k \setminus \{0\}} \frac{Q(g; D)}{|g|^2_{L^2(D)}} \leq \lambda_k(D) + C_k \operatorname{Cap} A^{1/2},$$

and we conclude the proof of Theorem 1.2.

Corollary 2.4. Given a closed subset $A \subset D$ with $\operatorname{Cap} A = 0$ let $\{V_j\}$ be the family of open sets given in Proposition 2.3 and let $\lambda_k(D \setminus V_j)$ be the k-th eigenvalue of the Dirichlet problem of the operator L in $D \setminus V_j$. Then

$$\lambda_k(D \setminus V_j) \to \lambda_k(D) \quad \text{as } j \to \infty.$$

In particular, if $\lambda_k(D \setminus V_j) \geq 0$ for all $j = 1, 2, \ldots$, then $\lambda_k(D) \geq 0$.

3. Proof of Theorem 1.1

We start by presenting an equivalent condition for r-stability. A simple computation shows that

$$\operatorname{trace}(B^2 P_r) = ||\sqrt{P_r} B||^2 := \sum_{i=1}^n \theta_i^2(r).$$

On the other hand, Lemma 2.1 of [2] says that

$$\operatorname{trace}(B^2 P_r) = S_1 S_{r+1} - (r+2) S_{r+2}.$$

Thus, for r-minimal immersions, the Jacobi operator can be written as

$$T_r = L_r + \operatorname{trace}(B^2 P_r) = L_r + ||\sqrt{P_r} B||^2.$$

Using integration by parts, we have that

$$\begin{aligned}
I_r(f, f) &= \int_D (\langle P_r \nabla f, \nabla f \rangle - ||\sqrt{P_r} B||^2 f^2) dM \\
&= \int_D (|\sqrt{P_r} \nabla f|^2 - ||\sqrt{P_r} B||^2 f^2) dM.
\end{aligned}$$

So, in the case $P_r > 0$, to check that a regular domain $D \subset M$ is r-stable we just need to show that the last term above is always nonpositive or nonnegative for all $f \in C_0^\infty(D)$. Similarly in the case $P_r < 0$.

Now we fix $\varepsilon > 0$ and denote $D^\varepsilon = D \setminus V^\varepsilon(A)$. Then, the Gauss map g is a local diffeomorphism on D^ε. Let $\varphi : g(D^\varepsilon) \to \mathbb{R}$ be the positive first eigenfunction of the spherical Laplacian $\widetilde{\Delta}$ on $g(D^\varepsilon)$, that is,

$$\begin{cases}
\widetilde{\Delta} \varphi + \lambda_1^\varepsilon \varphi = 0 & \text{in } g(D^\varepsilon), \\
\varphi > 0 & \text{in } g(D^\varepsilon), \\
\varphi = 0 & \text{on } \partial g(D^\varepsilon),
\end{cases}$$

where $\lambda_1^\varepsilon = \lambda_1(D^\varepsilon)$ is the first eigenvalue of $\widetilde{\Delta}$ on $g(D^\varepsilon)$. Recall that, since $D^\varepsilon \subset D$, we have $\lambda_1^\varepsilon \geq \lambda_1$.

In the following we will consider the pull back metric $\widetilde{s}$ by g on D^ε and denote by $\widetilde{\nabla}$ the gradient, by $[\ ]$ the norm of a vector, and by dS the volume element in this metric. By Lemma 2.9 in [1] one have $\widetilde{\nabla} f = B^{-2} \nabla f$, for smooth functions f

on M and a simple computation gives $[X] = |BX|$, for any tangent vector X. Also we point out that, in the metric $\widetilde{s}$, the Gauss map $g : D^\varepsilon \to S^n$ is a local isometry. Now let $\psi = \varphi \circ g$ defined in D^ε. Then ψ is positive and satisfies $\widetilde{\Delta}\psi + \lambda_1^\varepsilon \psi = 0$ in D^ε. Thus, by Corollary 1 in [6], we have that the first eigenfunction of the operator $\widetilde{\Delta} + \lambda_1^\varepsilon$ is nonnegative:

$$(3.1) \quad 0 \le \inf\left\{ \int_{D^\varepsilon} ([\widetilde{\nabla}f]^2 - \lambda_1^\varepsilon f^2)dS : f \in C_0^\infty(D^\varepsilon), \int_{D^\varepsilon} f^2 dS = 1\right\}.$$

Now, since $\det g = S_n$ and the immersion is r-special on D^ε, we have $dS = |S_n|dM = c||\sqrt{P_r}B||^2 dM$, where c is positive constant. Also, by hypothesis, $\lambda_1^\varepsilon \ge \lambda_1 \ge \tau$. Thus

$$(3.2) \quad \int_{D^\varepsilon} ([\widetilde{\nabla}f]^2 - \lambda_1^\varepsilon f^2)dS \le c\int_{D^\varepsilon} ([\widetilde{\nabla}f]^2 - \tau f^2)||\sqrt{P_r}B||^2 dM.$$

Observe now that $\rho_i = \frac{\theta_i^2(r)}{\sum_j \theta_j^2(r)}$, $i = 1, \ldots, n$, are the eigenvalues of the operator $\frac{\sqrt{P_r}B}{||\sqrt{P_r}B||}$. Therefore, by definition

$$\tau := \max_{i,D} \frac{1}{\rho_i^2} = \max_D \left|\left(\frac{\sqrt{P_r}B}{||\sqrt{P_r}B||}\right)^{-1}\right|^2.$$

So,

$$[\widetilde{\nabla}f]^2 \le \left|\left(\frac{\sqrt{P_r}B}{||\sqrt{P_r}B||}\right)^{-1}\right|^2 \left[\left(\frac{\sqrt{P_r}B\widetilde{\nabla}f}{||\sqrt{P_r}B||}\right)\right]^2 \le \tau\left[\left(\frac{\sqrt{P_r}B\widetilde{\nabla}f}{||\sqrt{P_r}B||}\right)\right]^2.$$

Using this on (3.2), we have

$$
\begin{aligned}
c\int_{D^\varepsilon} ([\widetilde{\nabla}f]^2 - \tau f^2)||\sqrt{P_r}B||^2 dM &\le c\tau\int_{D^\varepsilon} \left(\frac{[\sqrt{P_r}B\widetilde{\nabla}f]^2}{||\sqrt{P_r}B||^2} - f^2\right)||\sqrt{P_r}B||^2 dM \\
&= c\tau\int_{D^\varepsilon} \left([\sqrt{P_r}B\widetilde{\nabla}f]^2 - ||\sqrt{P_r}B||^2 f^2\right)dM.
\end{aligned}
$$

Now, we recall that $\widetilde{\nabla}f = B^{-2}\nabla f$ and $[X] = |BX|$ to find

$$[\sqrt{P_r}B\widetilde{\nabla}f]^2 = |\sqrt{P_r}\nabla f|^2.$$

We conclude that, for any $f \in C_0^\infty(D^\varepsilon)$ with $\int_{D^\varepsilon} f^2 dS = 1$,

$$0 \le c\tau\int_{D^\varepsilon} \left(|\sqrt{P_r}\nabla f|^2 - ||\sqrt{P_r}B||^2 f^2\right)dM.$$

This implies that the first eigenvalue of the Jacobi operator T_r is nonnegative on D^ε and thus, by Corollary 2.4, we have that first eigenvalue of the Jacobi operator T_r is nonnegative in all D. This shows that D is r-stable.

Remark: *In the case $P_r < 0$ we may set $Q_r = -P_r$ in the above computations to conclude that*

$$0 \le c\tau\int_{D^\varepsilon} \left(|\sqrt{Q_r}\nabla f|^2 - ||\sqrt{Q_r}B||^2 f^2\right)dM.$$

On the other hand the Jacobi operator is given by

$$
\begin{aligned}
T_r f &= \operatorname{div}(P_r\nabla f) + \operatorname{trace}(B^2 P_r) \\
&= -\operatorname{div}(Q_r\nabla f) + \operatorname{trace}(B^2 Q_r).
\end{aligned}
$$

We conclude that the first eigenvalue of T_r is nonpositive on D and therefore we obtain r-stability according to our definition.

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