Equilibrium States for Non-uniformly Expanding Maps

Krerley Oliveira *

May 4, 2004

Abstract

We construct equilibrium states, including measures of maximal entropy, for a large (open) class of non-uniformly expanding maps on compact manifolds. Moreover, we study uniqueness of these equilibrium states, as well as some of their ergodic properties.

1 Introduction

The theory of equilibrium states originates from statistical mechanics and was thoroughly developed, in the classical setting of uniformly hyperbolic dynamical systems, in the seventies and eighties, especially by Sinai, Ruelle, Bowen, Parry and Walters.

In general, given a continuous transformation $f : M \to M$ on a compact metric space, and given a continuous function ϕ, we call **equilibrium state** for (f, ϕ) a Borel probability measure μ_ϕ such that

$$h_{\mu_\phi}(f) + \int \phi d\mu_\phi = \sup_{\mu \in \mathcal{I}} \{h_\mu(f) + \int \phi d\mu\},$$

where the supremum is taken over the set $\mathcal{I}$ of f-invariant probabilities. That is, an equilibrium state is a maximum of the function $F_\phi : \mathcal{I} \to \mathbb{R}$ defined by

$$F_\phi(\mu) = h_\mu(f) + \int \phi d\mu.$$

It is now classical that for uniformly hyperbolic diffeomorphisms, as well as for uniformly expanding maps, equilibrium states always exist and they are unique if the potential is Hölder continuous, assuming that the transformation f is transitive. See [Par64, Sin72, Bow75, Rue89]. Moreover, the equilibrium

*The author is supported by CNPq, Brazil.

states coincide with the Gibbs measures, that is, the invariant probability measures satisfying

$$\mu(\mathcal{B}_\epsilon(n,x)) \cong \exp\,(\sum_{i=0}^{n-1} \phi(f^i(x)) - nP) \tag{1}$$

for some $P \in \mathbb{R}$, called the *pressure* of ϕ, where $\mathcal{B}_\epsilon(n,x)$ is the dynamical ball of length n and size ϵ around x,

$$\mathcal{B}_\epsilon(n,x) = \{y \in M; d(f^i(y), f^i(x)) \leq \epsilon, \text{ for every } 0 \leq i \leq n-1\},$$

and $\cong$ means equality up to a uniform factor, independent of x and n. In this setting, the pressure P is given by

$$P = \sup_{\mu \in \mathcal{I}} \{h_\mu(f) + \int \phi d\mu\}.$$

Several authors have been studying equilibrium states for non-hyperbolic systems: Bruin, Keller [BK98] and Denker, Urbanski [DU92, Urb98], for interval maps and rational functions on the sphere, and Buzzi, Maume, Sarig [Buz99, BMD02, BS, Sar03] and Yuri [Yur99, Yur00, Yur03], for countable Markov shifts and for piecewise expanding maps in one and higher dimensions, to mention just a few of the most recent works. Several of these papers, and particularly [DU92, Sar03, Yur99, Yur00, Yur03], consider systems with neutral periodic points, a setting of non-hyperbolic dynamics which has attracted a great deal of attention over the last years. Despite all these important contributions, it is fair to say that the theory of equilibrium states is very much incomplete outside the uniformly hyperbolic case.

The present work may be seen as a step towards obtaining such a theory in a general setting of *non-uniformly hyperbolic* systems (non-zero Lyapunov exponents). Indeed, we prove existence of equilibrium states for fairly general potentials and for a robust (open) class of non-uniformly expanding maps. This class will be defined precisely in the next section. Here we just mention one of its main features:

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| \leq -2c < 0 \tag{2}$$

for "most" points, including a full measure set relative to the equilibrium states μ_ϕ that we construct. We prove that equilibrium states do exist for every potential ϕ with small variation, that is, such that

$$\sup \phi - \inf \phi < K \tag{3}$$

for some convenient constant K (see Definition 2.2 and comments following it). As a consequence of (2), these measures are non-uniformly expanding, that is, μ_ϕ-almost every point has only positive Lyapunov exponents. Moreover, is possible to prove that μ_ϕ has a kind of weak Gibbs property (see [Yur00]) as in

(1), with $\cong$ meaning equality up to a factor with subexponential growth on the orbit of each x.

The basic strategy for the construction is to find a subset $\mathcal{K}$ of invariant probability measures which are expanding and such that μ-almost every point has infinitely many hyperbolic times, in the sense of [Alv00, ABV00]. We prove that $\nu \mapsto F_\phi(\nu)$ is upper-semicontinuous on $\mathcal{K}$ and there exist maximum on $\mathcal{K}$ of F_ϕ. Using (3) we check that the maximum obtained on $\mathcal{K}$ is really a maximum of F_ϕ over all invariant probabilities.

These arguments apply, in particular, when the potential ϕ is constant, in which case μ_ϕ maximizes the entropy:

$$h_{\mu_\phi}(f) = h_{top}(f).$$

However, in this case we can go much further. Using a different approach, via semi-conjugation to a one-sided subshift of finite type, we are able to prove that the maximal entropy measure is unique and a Markov measure. If f is topologically mixing, this measure is Bernoulli.

Closing this introduction, we mention some questions that are naturally raised by our results. The first one is to prove uniqueness or ,at least, finiteness of the equilibrium states for Hölder potentials in the general situation of (3), under topological transitivity. In this direction, in a forthcoming work the author proves existence and uniqueness of equilibrium states for an open class of local diffeomorphisms and for potentials with low variation satisfying a summability condition. Another question concerns the need of hypothesis (3) itself. Ongoing work indicates that equilibrium states do exist also for potentials with large variation, but they may have both positive and negative Lyapunov exponents: from the viewpoint of these measures the dynamics looks hyperbolic, rather than expanding.

2 Setting and Statements

We always consider a $C^{1+\alpha}$ local diffeomorphism $f : M^l \to M^l$, defined on a compact Riemannian manifold with dimension l. Let m be normalized Lebesgue measure on M. We suppose that f satisfies, for positive constants δ_0, β, δ_1, σ_1, and p, $q \in \mathbb{N}$,

(H1) There exists a covering $B_1, \dots, B_p, \dots, B_{p+q}$ of M such that every $f|B_i$ is injective and

- f is uniformly expanding at every $x \in B_1 \cup \dots \cup B_p$:
$$\|Df(x)^{-1}\| \leq (1+\delta_1)^{-1}.$$
- f is never too contracting: $\|Df(x)^{-1}\| \leq (1+\delta_0)$ for every $x \in M$.

(H2) f is everywhere volume-expanding: $|\det Df(x)| \geq \sigma_1$ with $\sigma_1 > q$.

Define
$$V = \{x \in M; \|Df(x)^{-1}\| > (1+\delta_1)^{-1}\}.$$

(H3) There exists a set $W \subset B_{p+1} \cup \cdots \cup B_{p+q}$ containing V such that

$$M_1 > m_2 \quad \text{and} \quad m_2 - m_1 < \beta$$

where m_1 and m_2 are the infimum and the supremum of $\log \| \det Df \|$ on V, respectively, and M_1 and M_2 are the infimum and the supremum of $\log \| \det Df \|$ on W^c, respectively.

Definition 2.1. The supremum of F_ϕ over the set all invariant probability is called the *pressure of* ϕ and will be denoted by $P(\phi)$

Definition 2.2. Given $\phi : M \to \mathbb{R}$ continuous, we say that ϕ has ρ*-low variation* if

$$\max_{x \in M} \phi(x) < P(\phi) - \rho h_{top}(f).$$

Remark 2.3. Note that this is an open condition on the potential, with respect to the C^0 topology. Note also that this is somewhat more general than condition (3) for K small enough: assuming K is less than $(1-\rho)h_{top}(f)$, if ϕ satisfies (3) then $\tilde{\phi} = \phi - \inf \phi$ has ρ-low variation potential; the conclusions of Theorem A are not affected if one replaces ϕ by $\tilde{\phi}$, because potentials that differ by a constant have the same equilibrium states.

Our first main result is

Theorem A. *Assume hypotheses (H1), (H2), (H3) hold, with δ_0 and β sufficiently small. Then, there exists ρ such that if ϕ is a continuous potential with ρ-low variation then ϕ has some equilibrium state. Moreover, these equilibrium states are hyperbolic measures, with all Lyapunov exponents bigger than some $c(\delta_1, \sigma_1, p, q) > 0$.*

For maximal entropy mesures we are able to say a lot more, under the following additional hypothesis:

(H4) There exists a Markov partition $\mathcal{R} = \{R_1, \ldots, R_d\}$ for f such that

- $\mathcal{R}$ is transitive: for any i, j there exists a k such that $f^k(R_i) \cap R_j \neq \emptyset$; For simplicity in the proofs we also assume that $W \subset R_1$.

We say that a system (f, μ) is Bernoulli (respectively Markov), if it is ergodically equivalent to a subshift of finite type endowed with a Bernoulli (respectively Markov) measure. See e.g. [Mañ87] for definitions.

Theorem B. *Assume hypotheses (H1), (H2), (H3), (H4) hold with δ_0 and β sufficiently small. Then there exists a unique invariant measure μ_{max} with $h_{\mu_{max}}(f) = h_{top}(f)$. This measure also satisfies,*

1. *all Lyapunov exponents of μ_{max} are larger than some $c(\delta_1, \sigma_1, p, q) > 0$;*
2. *(f, μ_{max}) is Markov and, if f is topologically mixing, it is Bernoulli.*

Important related results have been obtained by Yuri [Yur99, Yur00, Yur03], where she studies equilibrium states for very general Markov systems. On the other hand, our hypotheses are different and, to the best of our knowledge, our approach for proving Theorem A is new. For one thing, we do not assume existence of a *generating* Markov partition: instead, we construct a special partition and *prove* that it is generating for a carefully chosen class of measures (the set $\mathcal{K}$ mentioned in the Introduction). In general terms, we exploit the notion of hyperbolic times to deduce from the dynamical behaviour certain facts valid at almost every point, uniform versions of which are taken as hypotheses in Yuri's approach. For instance, we need no analogue of hypothesis (C5) in [Yur99]: in fact, for our examples in Section 3 the diameters of cylinders *do not* tend to zero. Besides, there a low variation potential may not satisfies the condition (C4) in [Yur99]. A combination of both viewpoints should lead to further progress in this area.

Acknowledgements: I am very thankful to my advisor Marcelo Viana for his exceptional advice and friendship. Warm thanks go also to A. Tahzibi, J. Bochi, C. Matheus, and A. Arbieto for suggestions and many fruitful discussions. I am indebted to IMPA and its staff for a fine working environment, and to CNPq for financial support.

3 Examples

In this section, we sketch the construction of a non-hyperbolic map f_0 that satisfies the conditions in theorems A and B above. It will be clear from this construction that these conditions hold, in fact, for every map f C^1-close to f_0.

We start by considering any Riemann manifold that supports an expanding map $g : M \to M$. For simplicity, choose $M = \mathbb{T}^n$ the n-dimensional torus, and g an endomorphism induced from a linear map with eigenvalues $\lambda_n > \cdots > \lambda_1 > 1$. Denote by $E_i(x)$ the eigenspace associated to the eigenvalue λ_i in T_xM.

Since g is expanding, it admits a transitive Markov partition $R_1, \ldots, R_d$ with arbitrarily small diameter. We may suppose that $g|R_i$ is injective for every $i = 1, \ldots, d$. Replacing g by a iterate if necessary, we may suppose that there exists a fixed point p_0 of g and, renumbering if necessary, this point is contained in the interior of the rectangle R_d of the Markov partition.

Considering a small neigbourhood $W \subset R_d$ of p_0 we deform g inside W along the direction E_1. This deformation consists essentially in rescaling the expansion along the invariant manifold associated to E_1 by a real function α. Let us be more precise:

Considering W small, we may identify W with a neighbourhood of 0 in $\mathbb{R}^n$ and p_0 with 0. Without loss of generality, suppose that $W = (-2\epsilon, 2\epsilon) \times B_{3r}(0)$, where $B_{3r}(0)$ is the ball or radius $3r$ and center 0 in $\mathbb{R}^{n-1}$. Consider a function $\alpha : (-2\epsilon, 2\epsilon) \to \mathbb{R}$ such $\alpha(x) = \lambda_1 x$ for every $|x| \geq \epsilon$ and for small constants γ_1, γ_2:

1. $(1 + \gamma_1)^{-1} < \alpha'(x) < \lambda_1 + \gamma_2$

2. $\alpha'(x) < 1$ for every $x \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$;

3. α is C^0-close to λ_1: $\sup\limits_{x\in(-\epsilon,\epsilon)} |\alpha(x) - \lambda_1 x| < \gamma_2$,

Also, we consider a bump function $\theta : B_{3r}(0) \to \mathbb{R}$ such $\theta(x) = 0$ for every $2r \leq |x| \leq 3r$ and $\theta(x) = 1$ for every $0 \leq |x| \leq r$. Suppose that $\|\theta'(x)\| \leq C$ for every $x \in B_{3r}(0)$. Considering coordinates $(x_1, \ldots, x_n)$ such that $\partial_{x_i} \in E_i$, define f_0 by:

$$f_0(x_1, \ldots, x_n) = (\lambda_1 x_1 + \theta(x_2, \ldots, x_n)(\alpha(x_1) - \lambda_1 x_1), \lambda_2 x_2, \ldots, \lambda_n x_n)$$

Observe that by the definition of θ and α we can extend f_0 smoothly to $\mathbb{T}^n$ as $f_0 = g$ outside W. Now, is not difficult to prove that f_0 satisfies the conditions (H1), (H2), (H3) and (H4) above.

First, we have that $\|Df_0(x)^{-1}\|^{-1} \geq \min\limits_{i=1,\ldots,n} \|\partial_{x_i} f_0\|$. Observe that:

$$\partial_{x_1} f_0(x_1, \ldots, x_n) = (\alpha'(x_1)\theta(x_2, \ldots, x_n) + (1 - \theta(x_2, \ldots, x_n))\lambda_1, 0, \ldots, 0)$$

$$\partial_{x_i} f_0(x_1, \ldots, x_n) = ((\alpha(x_1) - \lambda_1)\partial_{x_i}\theta(x_2, \ldots, x_n), 0, \ldots, \lambda_i, 0, \ldots, 0), \text{ for } i \geq 2.$$

Then, since $\|\partial_{x_i}\theta(x)\| \leq C$ for every $x \in B_{3r}(0)$, and $\alpha(x_1) - \lambda_1 x_1 \leq \gamma_2$ we have that $\|\partial_{x_i} f_0\| > (\lambda_i - \gamma_2 C)$ for every $i = 2, \ldots, n$. Moreover, by condition 1, $\|\partial_{x_1} f_0\| \leq \max\{\alpha'(x_1), \lambda_1\} \leq \lambda_1 + \gamma_2$, if we choose γ_2 small in such way that $\lambda_2 - \gamma_2 C > \lambda_1 + \gamma_2$ then:

$$\|\partial_{x_i} f_0\| > \|\partial_{x_1} f_0\|, \text{ for every } i \geq 2.$$

Notice also that $\|\partial_{x_1} f_0\| \geq \min\{\alpha'(x_1), \lambda_1\} \geq (1 + \gamma_1)^{-1}$. This prove that:

$$\|Df_0(x)^{-1}\|^{-1} \geq \min_{i=1,\ldots,n} \|\partial_{x_i} f_0\| \; (1 + \gamma_1)^{-1}.$$

Since f coincides with g outside W, we have $\|Df_0(x)^{-1}\| \leq \lambda_1^{-1}$ for every $x \in W^c$. Together with the above inequality, this proves condition (H1), with $\delta_0 = \gamma_1$.

Choosing γ_1 small and $p = d - 1$, $q = 1$, $B_i = R_i$ for every $i = 1, \ldots, d$, condition (H2) is imediate. Indeed, observe that the Jacobian of f_0 is given by the formula:

$$\det Df_0(x) = (\alpha'(x_1)\theta(x_2, \ldots, x_n) + (1 - \theta(x_2, \ldots, x_n))\lambda_1) \prod_{i=2}^{n} \lambda_i.$$

Then, if we choose $\gamma_1 < \prod_{i=2}^{n} \lambda_i - 1$:

$$\det Df_0(x) > (1+\gamma_1)^{-1} \prod_{i=2}^{n} \lambda_i > 1.$$

Therefore, we may take $\sigma_1 = (1+\gamma_1)^{-1} \prod_{i=2}^{n} \lambda_i > 1$.

To verify property (H3) for f_0, observe that if we denote by

$$V = \{x \in M; \|Df_0(x)^{-1}\| > (1+\delta_1)^{-1}\},$$

with $\delta_1 < \lambda_1 - 1$, then $V \subset W$. Indeed, since $\alpha(x_1)$ is constant equal to $\lambda_1 x_1$ outside W we have that $\|Df_0(x)^{-1}\| \leq \lambda_1^{-1} < (1+\delta_1)^{-1}$, for every $x \in W^c$. Given γ_3 close to 0, we may choose δ_1 close to 0 and α satisfying the conditions above in such way that,

$$\sup_{x,y \in V} \alpha'(x_1) - \alpha'(y_1) < \gamma_3.$$

If m_1 and m_2 are the infimum and the supremum of $|\det Df_0|$ on V, respectively,

$$m_2 - m_1 \leq C(\sup_{x,y \in V} \alpha'(x_1) - \alpha'(y_1)) < \gamma_3 C,$$

where $C = \prod_{i=2}^{n} \lambda_i$. Then, we may take $\beta = \gamma_3 C$ in (H3). If M_1 is the infimum of $|\det Df_0|$ on W^c, $M_1 > m_2$, since $\lambda_1 > (1+\delta_1) \geq \sup_{x \in V} \alpha'(x)$. Condition (H4) is clear from the construction, since $f_0 = g$ outside $W \subset R_d$, so the Markov property of $\{R_1, \ldots, R_d\}$ is not affected by the pertubation.

The arguments above show that the hypotheses $(H1), (H2), (H3)$ and $(H4)$ are satisfied by f_0. Moreover, if we one takes $\alpha(0) = 0$, then p_0 is fixed point for f_0, which is not a reppeler, since $\alpha'(0) < 1$. Therefore, f_0 is not a uniformly expanding map.

It is not difficult to see that this construction may be carried out in such way that f_0 does not satisfy the expansiveness property: there is a fixed hyperbolic saddle point p_0 such that the stable manifold of p_0 is contained in the unstable manifold of two other fixed points.

For a discussion of related examples see [Car93] and [BV00].

4 Expanding measures and hyperbolic times

The proof of Theorem A occupies this section and the next one. Let us begin by detailing a bit more our strategy to prove the existence of equilibrium states:

- To exhibit a subset $\mathcal{K}$ of invariant measures such that all their Lyapunov exponents are positive, and almost every point has infinitely many hyperbolic times.

- To show that there exists a common generating partition for all the measures in $\mathcal{K}$. This allows us to prove that the function $\mu \to h_\mu(f) + \int \phi d\mu$ is upper-semicontinuous on $\mathcal{K}$. Using this, we get that the maximum of $h_\mu(f) + \int \phi d\mu$ for measures μ in $\mathcal{K}$ is attained.
- To prove that if the potential has low variation, the maximum obtained over $\mathcal{K}$ is, in fact, a global maximum for the function $h_\mu(f) + \int \phi d\mu$ over all invariant measures.

Let us begin by stating our precise conditions on the constants δ_0 and β in the theorem. According to [ABV00, Appendix], if f satisfies (H1) then there exists $\gamma_0 < 1$ depending only on (σ_1, p, q) such that Lebesgue almost every point spends at most a fraction γ_0 of time inside $B_{p+1} \cup \cdots \cup B_{p+q}$.

Reducing δ_0 if necessary, we may find constants $\alpha > 0$, as close to 1 as we want, and $c > 0$ such that

$$(1+\delta_0)^\alpha (1+\delta_1)^{-(1-\alpha)} < e^{-2c} < 1. \tag{4}$$

We take $\alpha > \gamma_0$. By hypothesis (H3), $m_2 < M_1$ and $m_2 - m_1 < \beta$. So, taking β and δ_0 small, and α close enough to 1, we ensure that

$$\alpha m_2 + (1-\alpha) M_2 < \gamma_0 m_1 + (1-\gamma_0) M_1 - l \log(1+\delta_0). \tag{5}$$

Preparing the definition of $\mathcal{K}$, we introduce the compact convex set $K_\alpha \subset \mathcal{I}$ given by

$$K_\alpha = \{\mu \in \mathcal{I}; \mu(V) \leq \alpha\}.$$

Since Lebesgue almost every point spends at most a fraction $\gamma_0 < \alpha$ of time inside $V \subset B_{p+1} \cup \cdots \cup B_{p+q}$, all the ergodic absolutely continuous invariant measures constructed in [ABV00] belong to K_α. In particular, K_α is non-empty. We will see that K_α contains the equilibrium states of potentials with low variation.

Let us recall the ergodic decomposition theorem, as it is proven in [Mañ87]:

Theorem 4.1. *Given any invariant measure, there are ergodic invariant measures $\{\mu_x : x \in M\}$ depending measurably on the point x and such that*

$$\int f d\mu = \int (\int f d\mu_x) d\mu \quad \textit{for every } f \in L^1(d\mu).$$

Moreover, this decomposition $\{\mu_x : x \in M\}$ is essentially unique and, in fact, $\mu_x = \lim n^{-1} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ at μ-almost every point.

Our distinguished set of invariant measures is the non-empty set $\mathcal{K} \subset K_\alpha$ defined by

$$\mathcal{K} = \{\mu \in \mathcal{I}; \mu_x \in K_\alpha \text{ for } \mu\text{-a.e. } x\}$$

Definition 4.2. We say that a measure ν is *f-expanding with exponent c* if for ν-almost every $x \in M$ we have:

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -2c < 0.$$

Remark 4.3. If ν is an ergodic invariant measure, by the ergodic theorem the above definition is equivalent to $\int \log \|Df^{-1}\| d\nu \leq -2c$.

Remark 4.4. Given any ergodic invariant measure ν, the Lyapunov exponents of (f, ν) are all positive if and only if ν is f^N-expanding for some iterate N. See [ABV00]. Also, if every invariant measure ν is f-expanding, then f is a uniformly expanding map. See [AAS03]. In fact, the same conclusion holds, more generally, if all invariant measures have only positive Lyapunov exponents. See [Cao].

The next statement proves that all measures in $\mathcal{K}$ are f-expanding, with uniform exponent. Let $c > 0$ be as in (4).

Lemma 4.5. *Every measure in $\mu \in \mathcal{K}$ is f-expanding with exponent c:*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -2c < 0 \quad \textit{for } \mu\textit{-almost every } x \in M.$$

Proof. Suppose that $\mu \in \mathcal{K}$ is ergodic. Since $\mu \in K_\alpha$ and $\mu(V) \leq \alpha$, by the ergodic theorem almost every point spends a fraction less than α inside V: there exist an invariant set A with $\mu(A) = 1$ such that for every $x \in A$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_V(f^i(x)) \leq \alpha.$$

By hypothesis (H1), we have that $\|Df(y)^{-1}\| \leq (1 + \delta_0)$ for every $y \in V$. Moreover, $\|Df(y)^{-1}\| \leq (1 + \delta_1)^{-1}$ for $y \in V^c$. This implies that

$$\begin{aligned}\frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| &\leq \frac{1}{n} \log(1 + \delta_0)^{\alpha n}(1 + \delta_1)^{-(1-\alpha)n} \\ &\leq \log(1 + \delta_0)^{\alpha}(1 + \delta_1)^{-(1-\alpha)} < -2c < 0\end{aligned}$$

for all $x \in A$.

To finish , let H be the set of x that satisfy the condition in the conclusion of the lemma. Since every $\mu \in \mathcal{K}$ is $\mu = \{\mu_x\}$ a convex combination of ergodic measures μ_x in K_α, by the previous case, $\mu_a(H) = 1$ for μ almost every a, and this implies that $\mu(H) = \int \mu_x(H) d\mu = 1$. $\square$

Now we need the notion of hyperbolic time, first introduced by Alves [Alv00].

Definition 4.6. We say that n is a *hyperbolic time* for x with exponent c, if for every $1 \le j \le n$:

$$\prod_{k=0}^{j-1} \|Df(f^{n-k}(x))^{-1}\| \le e^{-cj}.$$

To prove that for an f-expanding measure almost every point admits infinitely many hyperbolic times, we need the following lemma due to Pliss. See for instance [ABV00] for a proof.

Lemma 4.7. *Given $A \ge c_2 > c_1 > 0$, let $d_0 = \frac{c_2 - c_1}{A - c_1}$. If $a_1, \dots, a_n$ are real numbers such that $a_i \le A$ and*

$$\sum_{i=1}^{n} a_i \ge c_2 n$$

then there are integer numbers $l > d_0 n$ and $1 < n_1 < \cdots < n_l \le n$ so that, for every $0 \le k \le n_i$ and $i = 1, \dots, l$:

$$\sum_{j=k+1}^{n_i} a_j \ge c_1(n_i - n)$$

Using this lemma, we get

Lemma 4.8. *For every invariant measure ν which is f-expanding with exponent c, there exists a full ν-measure set $H \subset M$ such that every $x \in H$ has infinitely many hyperbolic times $n_i = n_i(x)$ with exponent c and, in fact, the density of hyperbolic times at infinity is larger than some $d_0 = d_0(c) > 0$:*

1. $\displaystyle\prod_{k=0}^{j-1} \|Df^{-1}(f^{n_i - k}(x))\| \le e^{-cj}$ *for every* $0 \le j \le n_i$

2. $\displaystyle\liminf_{n\to\infty} \frac{\sharp\{0 \le n_i \le n\}}{n} \ge d_0 > 0.$

Proof. By the definition of f-expanding measures, there exists a set H with $\nu(H) = 1$ such that given any $x \in H$ we have

$$\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| \le -\frac{3c}{2} n$$

for every n large enough. Then, it suffices to take $A = \sup_{x\in M} -\log \|Df(x)^{-1}\|$, $c_1 = c$, $c_2 = \frac{3c}{2}$ and $a_i = -\log \|Df(f^{i-1}(x))^{-1}\|$ in the previous lemma. $\square$

The next lemma asserts that points at hyperbolic times have unstable manifolds with size uniformly bounded from below.

Lemma 4.9. *There exists $\epsilon_0 > 0$ such that for every x and n_i a hyperbolic time of x, if $z \in M$ satisfies $f^{n_i}(z) \in B_{\epsilon_0}(f^{n_i}(x))$ then*

$$d(f^{n_i-j}(x), f^{n_i-j}(z)) \leq e^{\frac{-c}{2}j} d(f^{n_i}(z), f^{n_i}(x)) \quad \textit{for every } 0 \leq j \leq n_i(x).$$

Proof. Since Df is uniformly continuous and a local diffeomorphism, there exists ϵ_0 such that for every $\xi, \eta \in M$ with $\xi \in B_{\epsilon_0}(\eta)$ then

$$\frac{\|Df(\xi)^{-1}\|}{\|Df(\eta)^{-1}\|} \leq e^{\frac{c}{2}}.$$

By definition, if n_i is hyperbolic time for x then $\prod_{k=0}^{j-1} \|Df(f^{n_i-k}(x))^{-1}\| \leq e^{-cj}$ for every $0 \leq j \leq n_i$. Observe that $d(f^{n_i-1}(z), f^{n_i-1}(x)) \leq \epsilon_0$. This is because $f^{n_i}(z) \in B_{\epsilon_0}(f^{n_i}(x))$ and, by the previous observation, the norm of the derivarive of the inverse branch of f that sends $f^{n_i}(x)$ to $f^{n_i-1}(x)$ is less than $e^{-\frac{c}{2}}$ restricted to $B_{\epsilon_0}(f^{n_i}(x))$. Arguing by induction,

$$\prod_{k=0}^{j-1} \|Df(f^{n_i-k}(z))^{-1}\| \leq e^{\frac{-c}{2}j} \quad \text{for all} \quad 0 \leq j \leq n_i.$$

We conclude that $d(f^{n_i-j}(x), f^{n_i-j}(z)) \leq e^{\frac{-c}{2}j} d(f^{n_i}(x), f^{n_i}(z))$, proving the lemma. $\square$

Since c is fixed, we will write simply ν-expanding to mean ν-expanding with exponent c.

5 Existence of equilibrium states for continuous low variation potentials

Our main aim in this section is to establish the existence of equilibrium states for continuous low variation potentials, in order to prove theorem A. In particular, this applies to $\phi = 0$, which always has low variation. Thus, our construction also yields maximal entropy measures for these transformations.

Beforehand, we use some results of the previous section to establish expansiveness for measures in $\mathcal{K}$.

Definition 5.1. Given $\epsilon > 0$ we define the set $A_\epsilon(x)$ by:

$$A_\epsilon(x) = \{y \in M; d(f^n(x), f^n(y)) \leq \epsilon \text{ for every } n \geq 0\}.$$

By definition, f is an **expansive map** with expansiveness constant $\tilde{\epsilon}$ if and only if $A_\epsilon(x) = \{x\}$ for every $x \in M$ and $\epsilon < \tilde{\epsilon}$.

Lemma 5.2. *Suppose that $\mu \in \mathcal{K}$ and let ϵ_0 be as constructed in lemma 4.9. Then for μ-almost every $x \in M$ and any $\epsilon < \epsilon_0$,*

$$A_\epsilon(x) = \{x\}.$$

Proof. By lemma 4.8 we have that almost every $x \in M$ has infinitely many hyperbolic times $n_i(x)$. By lemma 4.9, if $z \in A_\epsilon(x)$ with $\epsilon < \epsilon_0$ then for any n_i we have

$$d(x,z) \le e^{-\frac{c}{2}n_i} d(f^{n_i}(x), f^{n_i}(z)) \le e^{-\frac{c}{2}n_i}\epsilon.$$

Making $n_i \to \infty$ we deduce that $x = z$. □

Let $\mathcal{P} = \{P_1, \ldots, P_l\}$ be any partition of M in measurable sets with diameter less than ϵ_0. From the above lemma we get

Corollary 5.3. *$\mathcal{P}$ is a generating partition for every $\mu \in \mathcal{K}$.*

Proof. Define

$$\mathcal{P}^{(n)} = \{C^{(n)} = P_{i_0} \cap \cdots \cap f^{-n+1}(P_{i_{n-1}})\}, \quad \text{for each } n \ge 1.$$

We need to prove that given any measurable set A and given $\delta > 0$ there exist elements $C_1^{(n)}, \ldots, C_m^{(n)}$ of $\mathcal{P}^{(n)}$ such that

$$\mu(\bigcup C_i^{(n)} \Delta A) \le \delta.$$

Consider $K_1 \subset A$ and $K_2 \subset A^c$ compact sets such that $\mu(K_1 \Delta A) \le \delta$ and $\mu(K_2 \Delta A^c) \le \delta$. Let $r = d(K_1, K_2) > 0$. Lemma 5.2 gives that if n is big enough then $\operatorname{diam}\mathcal{P}^{(n)}(x) \le \frac{r}{2}$, for x in a set with μ-measure bigger than $1 - \delta$. Consider the sets $C_1^{(n)}, \ldots, C_m^{(n)} \in \mathcal{P}^{(n)}$ that intersect K_1. Then

$$\begin{aligned}\mu(\bigcup C_i^{(n)} \Delta A) &= \mu(\bigcup C_i^{(n)} - A) + \mu(A - \bigcup C_i^{(n)}) \\ &\le \mu(A - K_1) + \mu(A^c - K_2) + \delta \le 3\delta.\end{aligned}$$

This proves the claim. □

Remark 5.4. Recalling the definition of entropy with respect to a partition $\mathcal{Q}$,

$$H_\mu(\mathcal{Q}) = \sum_{Q \in \mathcal{Q}} -\mu(Q)\log\mu(Q),$$

we have that for any partition $\mathcal{Q}$ such $\mu_0(\partial Q) = 0$, for each $Q \in \mathcal{Q}$, then the function $\mu \to H_\mu(\mathcal{Q})$ is continuous in μ_0. This imply that

$$\mu \mapsto h_\mu(f, \mathcal{P}) = \inf_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P}^{(n)}).$$

is upper-semicontinuous in μ_0.

Observe that, by corollary 5.3, $\mathcal{P}$ is a generating partition for every $\mu \in \mathcal{K}$. So, as a consequence of Kolmogorov-Sinai's theorem(see e.g. [Mañ87]),

Corollary 5.5. *For every measure $\mu \in \mathcal{K}$ we have $h_\mu(f) = h_\mu(f, \mathcal{P})$.*

The next lemma will allow us prove that if ϕ has ρ-low variation, the function F_ϕ restrict to $\mathcal{K}$ has a maximum μ_ϕ and this maximum is in fact an equilibrium state for ϕ.

Lemma 5.6. *All ergodic measures η outside $\mathcal{K}$ have small entropy: there exists $\rho < 1$ such that if $\eta \in \mathcal{K}^c$ is ergodic then*

$$h_\eta(f) \leq \rho h_{top}(f).$$

Proof. As we are supposing that η is ergodic, we have that $\eta(V) > \alpha$, since every ergodic measure μ such $\mu(V) \leq \alpha$ lies in $\mathcal{K}$. Denoting $\lambda_1(x) \geq \lambda_2(x) \geq \dots \lambda_s \geq 0 > \lambda_{s+1} \dots \geq \lambda_l(x)$ the Lyapunov exponents in x, we know that $\lambda_i = \lambda_i(x)$ is constant η-almost everywhere. By the theorem of Oseledets [Ose68],

$$\int \log \| \det Df \| d\eta = \sum_{i=1}^{l} \lambda_i. \tag{6}$$

On the other hand, we have that $\lambda_l > -\log(1+\delta_0)$, since by hyphotesis, $\|Df(x)^{-1}\| \leq 1 + \delta_0$. By Ruelle's inequality (see [Rue78]) we have that

$$h_\eta(f) \leq \sum_{i=1}^{s} \lambda_i = \int \log \| \det Df(x) \| d\eta - \sum_{i=s+1}^{l} \lambda_i. \tag{7}$$

Since $m_2 = \sup_{x \in V} \log \| \det Df(x) \| < M_2 = \sup_{x \in V^c} \log \| \det Df(x) \|$ and $\eta(V) > \alpha$ we have:

$$\begin{aligned} h_\eta(f) \leq \int \log |\det Df| d\eta &\leq \eta(V) m_2 + (1 - \eta(V)) M_2 + (l - s) \log(1 + \delta_0) \\ &\leq \alpha m_2 + (1-\alpha) M_2 + l \log(1+\delta_0) \end{aligned}$$

Let μ_0 be any ergodic absolutely continuous invariant measure as constructed in [ABV00]. Since Lebesgue almost every point spends at most a fraction γ_0 of time inside $W \subset B_{p+1} \cup \dots \cup B_{p+q}$, we have that $\mu_0(W) < \gamma_0$. As f is $C^{1+\alpha}$ and μ_0 is absolutely continuous, we may use Pesin's entropy formula (see [Pes77]):

$$h_{\mu_0}(f) = \int \log \| \det Df \| d\mu_0 \geq \mu_0(W) m_1 + (1 - \mu_0(W)) M_1$$

As $\mu_0(W) \leq \gamma_0$ and $m_1 < M_1$, we conclude that

$$\gamma_0 m_1 + (1 - \gamma_0) M_1 \leq h_{\mu_0}(f).$$

By (5),

$$\alpha m_2 + (1-\alpha) M_2 < \gamma_0 m_1 + (1-\gamma_0) M_1 - l \log(1 + \delta_0).$$

So, we can choose $\rho < 1$ such that

$$\begin{aligned} h_\eta(f) &\leq \alpha m_2 + (1-\alpha) M_2 + l \log(1+\delta_0) < \rho(\gamma_0 m_1 + (1-\gamma_0) M_1) \\ &< \rho h_{\mu_0}(f) \leq \rho h_{top}(f). \end{aligned}$$

This proves lemma 5.6. $\square$

Remark 5.7. Observe that if μ_0 is some SRB measure for f, from the proof of previous lemma, we may choose $\rho < \frac{h_{\mu_0}}{h_{top}(f)}$.

Observe that it follows from the lemma 5.6 and the variational principle:

Corollary 5.8 (Variational Principle for expanding measures). *If ϕ is a ρ-low variation potential, then:*

$$\sup_{\nu \in \mathcal{K}} h_\nu(f) + \int \phi d\nu = P(\phi)$$

In particular,

$$\sup_{\nu \in \mathcal{K}} h_\nu(f) = h_{top}(f)$$

Proof. Denote by E the set of all ergodic invariant probabilities, to prove the lemma, we just need to prove that:

$$\sup_{\nu \in \mathcal{K}} F_\phi(\nu) = \sup_{\nu \in E} F_\phi(\nu) \tag{8}$$

Since

$$P(\phi) = \sup_{\nu \in E} F_\phi(\nu)$$

To prove 8, note that by lemma 5.6 we have that for $\nu \in \mathcal{K}^c$ ergodic, then $h_\nu(f) \le \rho h_{top}(f)$. This imply that:

$$F_\phi(\nu) = h_\nu(f) + \int \phi d\nu \le \rho h_{top}(f) + \max_{x \in M} \phi(x) < P(\phi)$$

and this prove the corollary.

$\square$

6 Proof of theorem A

First of all, observe that the previous corollary asserts that $\sup_{\nu \in \mathcal{K}} F_\phi(\nu) = P(\phi)$. To prove that there exists some equilibrium states, consider a sequence of measures $\mu_k \in \mathcal{K}$ such $F_\phi(\mu_k)$ converge to $P(\phi)$.

Without loss of generality, we suppose that $\mu_k \to \mu$ weakly. We prove that μ is an equilibrium state for ϕ and belongs to $\mathcal{K}$.

First, we claim that $F_\phi(\mu) = P(\phi)$. In fact, fix $\mathcal{P}$ a partition with diameter less than ϵ_0, such $\mu(\partial P) = 0$ for any $P \in \mathcal{P}$. Observe that, since $\mu_k \in \mathcal{K}$, we have that $h_{\mu_k}(f) = h_{\mu_k}(f, \mathcal{P})$, by corollary 5.5. Then:

$$P(\phi) = \sup_{\nu \in \mathcal{K}} F_\phi(\nu) = \limsup F_\phi(\mu_k) = \limsup h_{\mu_k}(f) + \int \phi d\mu_k$$

As, by remark 5.4, the function $\nu \to h_\nu(f,\mathcal{P})$ is upper-semicontinuous in μ, thus:

$$\limsup h_{\mu_k}(f,\mathcal{P}) + \int \phi d\mu_k \leq h_\mu(f,\mathcal{P}) + \int \phi d\mu \leq h_\mu(f) + \int \phi d\mu = F_\phi(\mu)$$

Then, we have

$$P(\phi) = \sup_{\nu \in \mathcal{K}} F_\phi(\nu) \leq F_\phi(\mu) \leq P(\phi),$$

which imply that μ is equilibrium state for ϕ.

Now, we prove that any measure η such $F_\phi(\eta) = \sup_{\nu \in \mathcal{K}} F_\phi(\nu)$ belongs to $\mathcal{K}$, proving that all equilibrium states belong to $\mathcal{K}$.

In fact, if $\eta = \{\eta_x\}$ is the ergodic decomposition of η, we should prove that the set $S = \{x \in M; \eta_x \in K_\alpha\}$ is a η-full measure set.

Using the fact that

$$h_\eta(f) = \int h_{\eta_x}(f)\, d\eta(x),$$

(see [Rok67], for instance), we have $F_\phi(\eta) = \int F_\phi(\eta_x)\, d\eta(x)$

Suppose by contradiction that $\eta(S^c) > 0$. Observe that if $y \in S^c$, then η_y is in the hyphoteses of lemma 5.6 and thus:

$$F_\phi(\eta_y) = h_{\eta_y} + \int \phi\, d\eta_y < \rho h_{top}(f) + \max_{x \in M} \phi(x) < P(\phi), \tag{9}$$

since ϕ is a ρ-low variation potential.

As for every $x \in S$ we have that $F_\phi(\mu_x) \leq P(\phi)$, the inequality 9 implies that $F_\phi(\eta) = \int F_\phi(\eta_x)\, d\eta(x) < P(\phi)$, which is a contradiction. Then, $\eta(S) = 1$ which imply, by the definition of $\mathcal{K}$ that $\eta \in \mathcal{K}$ and that all equilibrium states are in $\mathcal{K}$, completing the prove of theorem A.

7 Proof of Theorem B

We give a proof of the existence and uniqueness of a measure with maximum entropy. Throughout, we assume the additional hypothesis (H4): existence of a transitive Markov partition. Observe that we do not require this partition to be generating.

Firstly, by Theorem A, there exists some measure μ_{max} with maximal entropy. Our strategy to prove its unicity is transfer our problem to a subshift of finite type $\sigma^+ : \Sigma^+ \to \Sigma^+$ via ergodic conjugacy. If we denote $\partial R = \bigcup_{n=1}^{d} \partial R_i$ and $\tilde{M} = M - \bigcup_{n \geq 0} f^{-n}(\partial R)$, we have that if μ is an ergodic measure such $\mu(\partial\mathcal{R}) = 1$ then the entropy of μ is less than topological entropy, since $h_{top}(f|\partial\mathcal{R}) < h_{top}(f)$. Thus, we just consider invariant probabilities such

$\mu(\tilde{M}) = 1$. We may define a map $\Pi : \tilde{M} \to \Sigma^+$ over a subshift of finite type Σ^+ associate to some transition matrix A, by

$$\Pi(x) = (i_0, \dots, i_n, \dots) \text{ such that } f^n(x) \in P_{i_n}.$$

Observe that this map is a semiconjugacy between f and σ^+. Define the cylinders

$$[i] = [i_0, \dots, i_n] = \{x \in M; 0 \le j \le n, f^j(x) \in R_{i_j}\}.$$

Definition 7.1. Let (i_n) be the itinerary of x, defined by $f^n(x) \in R_{i_n}$, for each $n \ge 0$. We define $[x]$ to be the set

$$[x] = [i_0, ..., i_n, ...] = \{y \in M; f^n(y) \in R_{i_n}\}.$$

Given an invariant measure η satisfying $[x] = \{x\}$ for η-a.e., then Π is an ergodic conjugacy between (f, η) and $(\sigma^+, \Pi^\star\eta)$, where $\Pi^\star\eta)$ is defined by $\Pi^\star\eta)(A) = \eta(\Pi^{-1}(A))$. Observe that some measures can not be transported to the shift but, by lemma 5.6 any f-invariant measure with big entropy has hyperbolic times for almost everywhere. It allow us to prove $[x] = \{x\}$ for η-a.e., which imply that η is ergodically equivalent to some measure in the shift.

Using the classical fact that transitive subshifts of finite type have exactly one measure of maximal entropy, we prove that f admits only one measure μ_{max} with maximal entropy. If Σ^+ is topologically mixing, then its maximal measure is mixing ([Bow75]). Since, by Ornstein's Theorem([Mañ87]), every mixing Markov measure is Bernoulli, we have that μ_{max} is Bernoulli.

References

[Alv00] J. F. Alves. SRB measures for non-hyperbolic systems with multidimensional expansion. *Ann. Sci. École Norm. Sup.*, 33:1–32, 2000.

[AAS03] J. F. Alves, V. Araújo, and B. Saussol. On the uniform hyperbolicity of some nonuniformly hyperbolic systems. *Proc. Amer. Math. Soc.*, 131:1303–1309 (electronic), 2003.

[ABV00] J. F. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.*, 140:351–398, 2000.

[Bow75] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[BK98] H. Bruin and G. Keller. Equilibrium states for S-unimodal maps. *Ergod. Th. & Dynam. Sys.*, 18:765–789, 1998.

[Buz99] J. Buzzi. Markov extensions for multi-dimensional dynamical systems. *Israel J. Math.*, 112:357–380, 1999.

[BMD02] J. Buzzi and V. Maume-Deschamps. Decay of correlations for piecewise invertible maps in higher dimensions. *Israel J. Math.*, 131:203–220, 2002.

[BS] J. Buzzi and O. Sarig. Uniqueness of equilibrium states and multidimensional piecewise expanding maps. *Ergod. Th. & Dynam. Sys.* To appear.

[BV00] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[Cao] Y. Cao. Lyapunov exponents and uniform hyperbolicity. Preprint 2002.

[Car93] M. Carvalho. Sinai-Ruelle-Bowen measures for n-dimensional derived from Anosov diffeomorphisms. *Ergod. Th. & Dynam. Sys.*, 13:21–44, 1993.

[DU92] M. Denker and M. Urbański. The dichotomy of Hausdorff measures and equilibrium states for parabolic rational maps. In *Ergodic theory and related topics, III (Güstrow, 1990)*, volume 1514 of *Lecture Notes in Math.*, pages 90–113. Springer Verlag, 1992.

[Mañ87] R. Mañé. *Ergodic theory and differentiable dynamics*. Springer Verlag, 1987.

[Ose68] V. I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.

[Par64] W. Parry. On intrinsic Markov chains. *Trans. Amer. Math. Soc.*, 112:55–66, 1964.

[Pes77] Ya. B. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Russian Math. Surveys*, 324:55–114, 1977.

[Rok67] V.A. Rokhlin. Lectures on the entropy theory of measure-preserving transformations. *Russ. Math. Surveys*, 22 -5:1–52, 1967. Transl. from Uspekhi Mat. Nauk. 22 - 5 (1967), 3–56.

[Rue78] D. Ruelle. An inequality for the entropy of differentiable maps. *Bull. Braz. Math. Soc.*, 9:83–87, 1978.

[Rue89] D. Ruelle. The thermodynamical formalism for expanding maps. *Comm. Math. Phys.*, 125:239–262, 1989.

[Sar03] O. Sarig. Existence of Gibbs measures for countable Markov shifts. *Proc. Amer. Math. Soc.*, 131:1751–1758 (electronic), 2003.

[Sin72] Ya. Sinai. Gibbs measures in ergodic theory. *Russian Math. Surveys*, 27:21–69, 1972.

[Urb98] M. Urbański. Hausdorff measures versus equilibrium states of conformal infinite iterated function systems. *Period. Math. Hungar.*, 37:153–205, 1998. International Conference on Dimension and Dynamics (Miskolc, 1998).

[Yur99] M. Yuri. Thermodynamic formalism for certain nonhyperbolic maps. *Ergod. Th. & Dynam. Sys.*, 19:1365–1378, 1999.

[Yur00] M. Yuri. Weak Gibbs measures for certain non-hyperbolic systems. *Ergod. Th. & Dynam. Sys.*, 20:1495–1518, 2000.

[Yur03] M. Yuri. Thermodynamical formalism for countable to one markov systems. *Trans. Amer. Math. Soc.*, 335:2949–2971, 2003.

Krerley Oliveira (krerley@mat.ufal.br)
Departamento de Matematica - UFAL, Campus A.C. Simoes, s/n 57072-090
Maceio, Alagoas - Brazil