WELL-POSEDNESS FOR THE SCHRÖDINGER-DEBYE EQUATION

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Abstract. We establish local and global results for the initial value problem associated to the Schrödinger-Debye system for data in low regularity spaces. The main tool used is an optimal application of the Strichartz estimates for the linear Schrödinger operator. In the one dimensional case we also use Kato's smoothing effect to obtain global results in fractional Sobolev spaces.

1. Introduction

We study the Initial Value Problem (IVP) for the Schrödinger-Debye system

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = uv, \quad t \geq 0,\ x \in \mathbb{R}^n, \\ \tau\partial_t v + v = \epsilon|u|^2, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function, $v = v(x,t)$ is a real-valued function, $\tau > 0$, $\epsilon = \pm 1$ and Δ is the Laplacian operator in dimension n.

This system is derived from the Maxwell-Debye equations

$$\begin{cases} i\partial_t A + \frac{c}{2k\eta_0}\Delta A = \frac{w_0}{\eta_0}\nu A, \\ \tau\partial_t \nu + \nu = \eta_2|A|^2, \end{cases}$$

via the rescaling

$$u(x,t) = \sqrt{\frac{w_0|\eta_2|}{\eta_0}}\, A\left(\sqrt{\frac{c}{k\eta_0}}\, x,\ t\right),$$
$$v(x,t) = \frac{w_0}{\eta_0}\, \nu\left(\sqrt{\frac{c}{k\eta_0}}\, x,\ t\right).$$

The Maxwell-Debye system describes the non resonant delayed interaction of an electromagnetic wave with a media. In these equations A denotes the envelope of a light wave that goes through a media which response is non resonant. This wave induces a change ν of refractive index in the material (initially η_0 for an electromagnetic wave of frequency w_0) with a slight delay τ. The magnitude and the sign of the nonlinear coupling of the matter with the wave is described by the parameter η_2. The light velocity in the vacuum is denoted by c and k denotes the wave

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vector of the incident electromagnetic wave. See Newell and Moloney [13] for a more complete discussion of this model.

We can simplify the system (1.1) by writing explicitly the solution of

$$v(t) = e^{-t/\tau} v_0(x) + \tfrac{\epsilon}{\tau} \int_0^t e^{-(t-t')/\tau} |u(t')|^2 \, dt' \tag{1.2}$$

to obtain the decoupled integro-differential equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = e^{-t/\tau} u v_0(x) + \frac{\epsilon}{\tau} u \int_0^t e^{-(t-t')/\tau} |u(t')|^2 dt', & x \in \mathbb{R}^n, \ t \geq 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.3}$$

where $\tau > 0$ and $\epsilon = \pm 1$.

Using the integral formulation of this equation we have

$$u(t) = S(t)u_0 - i \int_0^t S(t-t')\big(F_0(u(t')) + F_1(u(t'))\big) \, dt', \tag{1.4}$$

where

$$S(t)u_0 = e^{\frac{it}{2}\Delta} u_0 = (e^{\frac{it}{2}|\xi|^2} \widehat{u}_0)^{\vee} \tag{1.5}$$

is the unitary group associated to the linear Schrödinger equation and

$$F_0(u(t)) = e^{-t/\tau} u v_0(x), \quad F_1(u(t)) = \tfrac{\epsilon}{\tau} u \int_0^t e^{-(t-t')/\tau} |u(t')|^2 dt'. \tag{1.6}$$

Previous results concerning well-posedness for the IVP (1.1) were obtained by Bidégaray in [2, 3]. We summarize them as follows:

(a_1) *local well-posedness in* $L^2(\mathbb{R}^n)$, *for data* $(u_0, v_0) \in L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$,

(a_2) *local well-posedness in* $H^1(\mathbb{R}^n)$, *for data* $(u_0, v_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$,

which are valid in dimensions $n = 1, 2, 3$. Here $H^s(\mathbb{R})$ denotes the standard L^2-based Sobolev space of order s.

We note that in these results the persistence property of the solutions was not obtained.

To prove these results the author used Strichartz estimates and a fixed point argument.

In [2] it was also shown that as τ tends to zero, solutions to the system (1.1) converge (in H^r with $r > 2 + n/2$) to those of the cubic nonlinear Schrödinger equation (NLS), namely

$$i\partial_t u + \tfrac{1}{2}\Delta u = \epsilon |u|^2 u, \tag{1.7}$$

at least on a certain time interval and for compatible initial data $v_0 = \epsilon |u_0|^2$.

Since local and global well-posedness in $H^s(\mathbb{R})$, $s \geq 0$, has been established for the NLS (1.7) (see [4, 7, 15]), in the one-dimensional, it is expected that the Cauchy problem for the equation (1.3) will be locally well-posed in $H^s(\mathbb{R})$ for $s \geq 0$, for given data v_0 in appropriate Sobolev space $H^k(\mathbb{R})$.

Our goal in this paper is to improve many of the results in [2, 3] and obtain new ones.

Regarding the well-posedness for the IVP (1.1) we will show the following results:

(b_1) *local and global well-posedness in* $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$,
(b_2) *local and global well-posedness in* $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$,

in dimensions $n=1,2,3$. Moreover, for the one-dimensional case, we will also show

(b_3) *local well-posedness for data in* $H^s(\mathbb{R})\times H^s(\mathbb{R})$, *for* $0<s<1$,
(b_4) *local and global well-posedness for data in* $H^{1/2}(\mathbb{R})\times L^2(\mathbb{R})$,
(b_5) *local and global well-posedness for data in* $H^s(\mathbb{R})\times H^k(\mathbb{R})$ *with* $1/2\leq s\leq 1$ *and* $s-1/2<k\leq s$.

To obtain our local results in the case $(b_1),(b_2)$ and (b_3) we will use the so called L^p-L^q or Strichartz estimates. These type of estimates were first established by Strichartz [14] for solutions of the linear Schrödinger equation. Generalizations of these estimates have been obtained by several authors. For instance, Ginibre and Velo [7] and Kenig, Ponce and Vega [10]. In the one dimension we will also use commutator estimates deduced by Kenig, Ponce and Vega in [12].

We proceed as follows. Instead of working with the system (1.1) we use its equivalent integral form (1.4). Then we use the L^p-L^q estimates, and the commutator estimates in the one-dimensional case, to show our results via the contraction mapping principle

Using that the L^2-norm for the solution u of the system (1.1) is conserved, i.e,

$$\int |u(x,t)|^2dx=\int |u_0(x)|^2dx, \tag{1.8}$$

we extend the local result in $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, in the case (b_1), to any time interval $[0,T]$. For the global result, in the case (b_2), we obtain a "priori" estimate in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ for the solution using the global result in $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$.

To obtain our global results in the cases (b_4) and (b_5), the main tool used is the smoothing effect obtained by Kenig, Ponce and Vega [11, 12], that is,

$$\|D_x^{1/2}\,S(t)u_0\|_{L^\infty_xL^2_T}\leq C\,\|u_0\|_{L^2_x}. \tag{1.9}$$

We use (1.9) to prove our local results in these cases. Using that the solutions obtained satisfy the integral equation (1.4) combined with the conservation law (1.8) we obtain a "priori" estimates for the local solutions, which are used to extend these to any time interval $[0,T]$.

Remark 1.1. *The method of proof used in* [2, 3] *does not yield the persistence of* $v(t)$ *in* H^1 *in the case* (a_2) *for* $n=2,3$ *or* $v(t)$ *in* L^∞ *in the case* (a_1). *Similarly, as in* [2, 3], *we perform a fixed-point procedure only on* u *to prove our results, but unlike these references we guarantee the persistence of the solution* $v(t)$ *in the same space where we take the initial data* v_0.

This paper is organized as follows. The statements of the main results will be given in Section 2. In Section 3 we prepare some preliminary results useful in the proofs of the main results. Section 4 will be devoted to establish local and global wellposedness for data in $L^2\times L^2$. The global well posedness result for data in fractional Sobolev spaces in the one dimensional case will be proved in Section 5. Finally, in Section 6 we show the global result for data in $H^1\times H^1$. Before leaving this section we introduce some notation.

In this work we use the following notations:

- The Fourier transform of f will be denoted by
$$\hat{f}(\xi) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$
- The Riesz potential of order $-s$ is denoted by
$$D_x^s f = c_s\, (|\xi|^s\, \widehat{f}(\xi))^{\vee},$$
- The $L^p - L^q$ norms are denoted as
$$\|f\|_{L^p_T L^q_x} = \left(\int_0^T \|f(\cdot,t)\|^p_{L^q} dt \right)^{1/p}$$
and
$$\|f\|_{L^p_x L^q_T} = \left(\int_{\mathbb{R}^n} \left(\int_0^T |f(\cdot,t)|^q\, dt \right)^{p/q} dx \right)^{1/p}.$$

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2. Main Results

In this section we present the statement of the main results in this paper. We need the following definition.

Definition 2.1. *Let $n \in \mathbb{N}$. The pair (r,q) is an admissible pair if satisfies*
$$\frac{2}{r} = n\left(\frac{1}{2} - \frac{1}{q}\right) \tag{2.1}$$
with
$$\begin{cases} 2 \le q \le \infty & if \quad n = 1, \\ 2 \le q < \infty & if \quad n = 2, \\ 2 \le q < \frac{2n}{n-2} & if \quad n \ge 3. \end{cases} \tag{2.2}$$

Theorem 2.1. *Let $n = 1,2,3$. Given $u_0, v_0 \in L^2(\mathbb{R}^n)$ there exist $T = T(\tau, \|u_0\|_{L^2}, \|v_0\|_{L^2})$ positive and a unique solution u of the IVP (1.3) satisfying*
$$u \in C\big([0,T] : L^2(\mathbb{R}^n)\big) \tag{2.3}$$
and
$$\|u\|_{L^r_T L^q_x} < \infty, \qquad \textit{for any admissible pair } (r,q). \tag{2.4}$$
Moreover, the map $(u_0, v_0) \longmapsto u(t)$ from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into the class defined by (2.3)–(2.4) is locally Lipschitz. In addition, from (2.3)–(2.4) one has that
$$v \in C\big([0,T] : L^2(\mathbb{R}^n)\big). \tag{2.5}$$
Furthermore, the above solution can be extended to any time interval $[0,T]$.

Theorem 2.2. *Let $n = 1, 2, 3$. Given $u_0, v_0 \in H^1(\mathbb{R}^n)$ there exist $T = T(\tau, \|u_0\|_{H^1}, \|v_0\|_{H^1})$ positive and a unique solution u of the IVP (1.3) satisfying*

$$u \in C\big([0,T] : H^1(\mathbb{R}^n)\big) \tag{2.6}$$

and

$$\|u\|_{L^r_T L^q_x} + \|\nabla u\|_{L^r_T L^q_x} < \infty, \quad \textit{for any admissible pair } (r,q). \tag{2.7}$$

Moreover, the map $(u_0, v_0) \longmapsto u(t)$ from $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ into the class defined by (2.6)–(2.7) is locally Lipschitz.

From (2.6)–(2.7) we also have that

$$v \in C\big([0,T] : H^1(\mathbb{R}^n)\big), \tag{2.8}$$

Furthermore, the above solution can be extended to any time interval $[0,T]$.

Now we let $s \in (0,1)$. Then we obtain the following results concerning well-posedness for the one-dimensional case in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$.

Theorem 2.3. *Let $0 < s < 1$. Then for any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ there exist $T = T(\tau, \|u_0\|_s, \|v_0\|_s) > 0$ and a unique solution u of the IVP (1.3) such that for $q \in [2, \infty]$*

$$u \in C\big([0,T] : H^s(\mathbb{R})\big) \tag{2.9}$$

and

$$\|u\|_{L^r_T L^q_x} < \infty, \quad \textit{with} \quad 2/r = 1/2 - 1/q. \tag{2.10}$$

Moreover, the map $(u_0, v_0) \longmapsto u(t)$ from $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ into (2.9)–(2.10) is locally Lipschitz.

From (2.9)–(2.10) one has that

$$v \in C\big([0,T] : H^s(\mathbb{R})\big), \tag{2.11}$$

In the particular case, when $1/2 \le s \le 1$, we can extend the local results in Theorem 2.3 for data v_0 less regular. We also show that the solutions obtained in this case are global. More precisely, if we define

$$I_s = \begin{cases} [0, 1/2] & \text{if} \quad s = 1/2, \\ (s - 1/2, s] & \text{if} \quad s \in (1/2, 1], \end{cases} \tag{2.12}$$

we have the following results

Theorem 2.4. *Let $s \in [1/2, 1]$. Then for any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^k(\mathbb{R})$ with $k \in I_s$ there exist $T = T(\tau, \|u_0\|_s, \|v_0\|_k) > 0$ and a unique solution u of the IVP (1.3) such that for $q \in [2, \infty]$*

$$u \in C\big([0,T] : H^s(\mathbb{R})\big) \tag{2.13}$$

and

$$\|u\|_{L^r_T L^q_x} + \|\partial_x u\|_{L^\infty_x L^2_T} < \infty, \quad \textit{with} \quad 2/r = 1/2 - 1/q. \tag{2.14}$$

Moreover, the map $(u_0, v_0) \longmapsto u(t)$ from $H^s(\mathbb{R}) \times H^k(\mathbb{R})$ into (2.13)–(2.14) is locally Lipschitz and

$$v \in C\big([0,T] : H^k(\mathbb{R})\big). \tag{2.15}$$

Corollary 2.1. *The solutions given by Theorem 2.4 extend to any time interval [0,T]. Moreover, when $s = 1/2$ and $0 \le k \le 1/2$ there are positive constants C, β, and γ such that*

$$\begin{aligned} &\|u\|_{L^\infty_T H^{1/2}_x} \le C\,\beta^{\gamma T}, \\ &\|v\|_{L^\infty_T H^k_x} \le \|v_0\|_{H^k} + \tfrac{C}{\tau}\, T^{3/4} \beta^{\gamma T}. \end{aligned} \tag{2.16}$$

3. Preliminary Estimates

In this section we collect known results on smoothing effect estimates of free Schrödinger evolution group.

First we consider the (IVP) associated to the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = 0, \quad x \in \mathbb{R}^n,\ t \ge 0, \\ u(x,0) = u_0(x), \end{cases} \tag{3.1}$$

whose solution is given by $u(x,t) = S(t)u_0(x)$ and $S(t)$ defined in (1.5).

We use the following well-known Strichartz estimates.

Proposition 3.1. *If (r_1, q_1) and (r_2, q_2) are admissible, then we have the following estimates*

$$\|S(t)u_0\|_{L^{r_1}_T L^{q_1}_x} \le C\|u_0\|_{L^2}, \tag{3.2}$$

$$\| \int_0^t S(t-t')G(\cdot,t')dt' \|_{L^{r_1}_T L^{q_1}_x} \le C\|G\|_{L^{r_2'}_T L^{q_2'}_x} \tag{3.3}$$

and

$$\| \int_0^t S(t-t')G(\cdot,t')dt' \|_{L^{r_1}_T L^{q_1}_x} \le C T^{(1/r_2'-1/2)} \|G\|_{L^{q_2'}_x L^2_T}, \quad \textit{for} \quad n = 1, 2, \tag{3.4}$$

where $\frac{1}{r_2} + \frac{1}{r_2'} = 1$ *and* $\frac{1}{q_2} + \frac{1}{q_2'} = 1$.

Remark 3.1. *The estimates (3.2) and (3.3) hold for all dimensions (n). The last estimate (3.4) is a slight modification of the (3.3) in the cases $n = 1, 2$.*

Proof. See Ginibre-Velo [8] for the proof of the first estimate (3.2). For the proof of (3.3) we can see Ginibre-Velo [8], K. Yajima [16], Cazenave-Weissler [4] and Kato [9]. The last estimate (3.4) is an immediate consequence of the (3.3). Indeed, we note that

$$\begin{cases} q_2 \in [2,\infty] \Longrightarrow r_2 \ge 4/n \Longrightarrow r_2' \le 4/(4-n) \le 2,\ (n = 1, 2), \\ q_2 \ge 2 \Longrightarrow 2 \ge q_2'. \end{cases}$$

Then, using Hölder's and Minkowski's inequalities we have

$$\|G\|_{L^{r_2'}_T L^{q_2'}_x} \le T^{(1/r_2'-1/2)} \|G\|_{L^2_T L^{q_2'}_x} \le T^{(1/r_2'-1/2)} \|G\|_{L^{q_2'}_x L^2_T},$$

and hence, (3.4) follows from (3.3). □

In the one dimensional case the main tool to obtain global results is the smoothing effect of Kato type for solutions of the linear Schrödinger equation. More precisely,

Proposition 3.2.

$$\sup_{x\in\mathbb{R}}\Big(\int_{-\infty}^{\infty}|D_x^{1/2}S(t)u_0(x)|^2dt\Big)^{1/2}\le C\|u_0\|_{L^2}, \tag{3.5}$$

$$\|D_x^{1/2}\int_0^t S(t-t')G(x,t')\,dt'\|_{L^2_x}\le C\,\|G\|_{L^1_xL^2_t} \tag{3.6}$$

and

$$\sup_{x\in\mathbb{R}}\Big(\int_{-\infty}^{+\infty}\Big|\partial_x\int_0^t S(t-t')G(x,t')\,dt'\Big|^2dt\Big)^{1/2}\le C\,\|G\|_{L^1_xL^2_t}. \tag{3.7}$$

These estimates were established by Kenig, Ponce and Vega [10, 11].

Proposition 3.3. *For any $\theta\in[0,1]$, we have*

$$\|D_x^{\theta/2}\int_0^t S(t-t')G(\cdot,t')dt'\|_{L^\infty_TL^2_x}\le C\,T^{(1-\theta)/2}\|G\|_{L_x^{2/(1+\theta)}L^2_T}. \tag{3.8}$$

Proof. The estimate (3.8) follows by interpolation (see [1]). □

Next we will prove a series of estimates for the nonlinear terms F_0 and F_1 defined in (1.6). Before doing so we will remind some useful results for our purpose.

To handle the nonlinear terms with fractional derivatives, we need the following commutator estimates deduced by Kenig, Ponce and Vega in [12].

Proposition 3.4. *Let $\alpha\in(0,1)$, $\alpha_1,\alpha_2\in(0,\alpha)$, $\alpha_1+\alpha_2=\alpha$ and $p,p_1,p_2,q,q_1,q_2\in(1,\infty)$ with $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$ and $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Then we have*

$$\|D_x^\alpha(fg)-fD_x^\alpha g-gD_x^\alpha f\|_{L^p_xL^q_T}\le C\|D_x^{\alpha_1}f\|_{L^{p_1}_xL^{q_1}_T}\|D_x^{\alpha_2}g\|_{L^{p_2}_xL^{q_2}_T}. \tag{3.9}$$

Moreover, (3.9) also holds when $q=1$ and when $(p,q)=(1,2)$.

$$\|D_x^\alpha(fg)-fD_x^\alpha g-gD_x^\alpha f\|_{L^p_xL^2_T}\le C\|f\|_{L^{p_1}_xL^\infty_T}\|D_x^\alpha g\|_{L^{p_2}_xL^2_T}. \tag{3.10}$$

$$\|D_x^\alpha(fg)-fD_x^\alpha g-gD_x^\alpha f\|_{L^p_x}\le C\|f\|_{L^\infty_x}\|D_x^\alpha g\|_{L^p_x}. \tag{3.11}$$

Proof. See Appendix in [12]. □

Finally we prove the following proposition which will be useful in the proof of Theorems 2.3 and 2.4.

Proposition 3.5. *Let $F_1(u)$ define as in (1.6), i.e,*

$$F_1(u(t))=\tfrac{\epsilon}{\tau}u\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'.$$

The following statements hold

(i) *For $s\in(0,1)$ we have*

$$\|D_x^sF_1(u)\|_{L^1_TL^2_x}\le\tfrac{C}{\tau}T^{3/2}\|u\|^2_{L^4_TL^\infty_x}\|u\|_{L^\infty_TH^s_x}. \tag{3.12}$$

(ii) *For $s \in [1/2, 1]$ we have*

$$\|D_x^{s-1/2} F_1(u)\|_{L_x^1 L_T^2} \le \frac{C}{\tau}\big(T^{5/4} + T^{3/2}\big)\|u\|^2_{L_T^\infty H_x^{s-1/2}}\big(\, \|u\|_{L_T^\infty H_x^s} + \|u\|_{L_T^4 L_x^\infty}\big). \tag{3.13}$$

Proof. For the case (i) the norm $\|D_x^s F_1(u)\|_{L_T^1 L_x^2}$ can be divided into three terms:

$$\|D_x^s F_1(u)\|_{L_T^1 L_x^2} \le \tfrac{1}{\tau}(A_{11} + A_{12} + A_{13})$$

where

$$\begin{aligned}
A_{11} = \big\| D_x^s\big(u \int_0^t e^{-(t-t')/\tau}|u(t')|^2 dt'\ \big) &- u\int_0^t e^{-(t-t')/\tau} D_x^s(|u(t')|^2)dt' \\
&- \big(D_x^s u\big)\int_0^t e^{-(t-t')/\tau}|u(t')|^2 dt'\ \big\|_{L_T^1 L_x^2},
\end{aligned}$$

$$A_{12} = \big\| u\int_0^t e^{-(t-t')/\tau} D_x^s(|u(t')|^2)dt'\ \big\|_{L_T^1 L_x^2},$$

$$A_{13} = \big\| \big(D_x^s u\big)\int_0^t e^{-(t-t')/\tau}|u(t')|^2 dt'\ \big\|_{L_T^1 L_x^2}.$$

For the first term we apply Proposition 3.4-(3.11) and Hölder's inequality in the time variable to get

$$\begin{aligned}
A_{11} &\le C\|u\|_{L_T^4 L_x^\infty}\|\int_0^t e^{-(t-t')/\tau} D_x^s(|u(t')|^2)dt'\|_{L_T^{4/3} L_x^2} \\
&\le CT^{3/4}\|u\|_{L_T^4 L_x^\infty}\|\int_0^t e^{-(t-t')/\tau} D_x^s(|u(t')|^2)dt'\|_{L_T^\infty L_x^2} \\
&\le CT^{3/4}\|u\|_{L_T^4 L_x^\infty}\|D_x^s(|u|^2)\|_{L_T^1 L_x^2} \\
&\le CT^{3/4}\|u\|_{L_T^4 L_x^\infty}\|u\|_{L_T^4 L_x^\infty}\|D_x^s u\|_{L_T^{4/3} L_x^2} \\
&\le CT^{3/2}\|u\|^2_{L_T^4 L_x^\infty}\|u\|_{L_T^\infty H_x^s}.
\end{aligned}$$

A similar argument shows that

$$A_{12} \le CT^{3/2}\|u\|^2_{L_T^4 L_x^\infty}\|u\|_{L_T^\infty H_x^s}.$$

For the last term using Hölder's inequality we obtain

$$\begin{aligned}
A_{13} &\le C\|D_x^s u\|_{L_T^1 L_x^2}\|\int_0^t e^{-(t-t')/\tau}|u(t')|^2 dt'\|_{L_T^\infty L_x^\infty} \\
&\le CT\|u\|_{L_T^\infty H_x^s}\||u|^2\|_{L_T^1 L_x^\infty} \\
&\le CT\|u\|_{L_T^\infty H_x^s}T^{1/2}\||u|^2\|_{L_T^2 L_x^\infty} \\
&\le CT^{3/2}\|u\|_{L_T^\infty H_x^s}\|u\|^2_{L_T^4 L_x^\infty}.
\end{aligned}$$

Then the above estimates for A_{11}, A_{12} and A_{13} give the desired estimate (3.12).

Next we show the case (ii). First we consider the particular case $s = 1/2$ and we get

$$
\begin{aligned}
\|F_1(u)\|_{L^1_xL^2_T} &\le \tfrac{C}{\tau}\|u\|_{L^2_xL^2_T}\Big\|\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'\ \Big\|_{L^2_xL^\infty_T}\\
&\le \tfrac{C}{\tau}T^{1/2}\|u\|_{L^\infty_TL^2_x}\|\overline{u}u\|_{L^1_TL^2_x}\\
&\le \tfrac{C}{\tau}T^{1/2}\|u\|_{L^\infty_TL^2_x}\|u\|_{L^{4/3}_TL^2_x}\|u\|_{L^4_TL^\infty_x}\\
&\le \tfrac{C}{\tau}T^{5/4}\|u\|^2_{L^\infty_TL^2_x}\|u\|_{L^4_TL^\infty_x}.
\end{aligned}
$$

Now for $1/2 < s \le 1$ we have

$$\|D_x^{s-1/2}F_1(u)\|_{L^1_xL^2_T} \le \tfrac{1}{\tau}(A_{21}+A_{22}+A_{23})$$

with

$$
\begin{aligned}
A_{21} = \Big\|D_x^{s-1/2}\Big(\,u\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'\,\Big)-u\int_0^t e^{-(t-t')/\tau}D_x^{s-1/2}(|u(t')|^2)dt'\\
-\big(D_x^{s-1/2}u\big)\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'\ \Big\|_{L^1_xL^2_T},
\end{aligned}
$$

$$A_{22} = \Big\|\big(D_x^{s-1/2}u\big)\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'\ \Big\|_{L^1_xL^2_T},$$

$$A_{23} = \Big\|u\int_0^t e^{-(t-t')/\tau}D_x^{s-1/2}(|u(t')|^2)dt'\ \Big\|_{L^1_xL^2_T}.$$

Proposition 3.4-(3.9), Minkowski's, Hölder's and Sobolev's inequalities yield

$$
\begin{aligned}
A_{21} &\le C\big\|D_x^{s/2-1/4}u\big\|_{L_x^{4/(3-2s)}L_T^{4/(3-2s)}}\Big\|D_x^{s/2-1/4}\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'\ \Big\|_{L_x^{4/(1+2s)}L_T^{4/(2s-1)}}\\
&\le CT^{(2s-1)/4}\big\|D_x^{s/2-1/4}u\big\|_{L_T^{4/(3-2s)}L_x^{4/(3-2s)}}\Big\|\int_0^t e^{-(t-t')/\tau}D_x^{s/2-1/4}(|u(t')|^2)dt'\ \Big\|_{L_x^{4/(1+2s)}L_T^\infty}\\
&\le CT^{(2s-1)/4}\big\|D_x^{s/2-1/4}u\big\|_{L_T^{4/(3-2s)}L_x^{4/(3-2s)}}\big\|D_x^{s/2-1/4}(|u|^2)\ \big\|_{L_x^{4/(1+2s)}L_T^1}\\
&\le CT^{1/2}\big\|D_x^{s/2-1/4}u\big\|_{L_T^\infty L_x^{4/(3-2s)}}\Big(\,2\|u\|_{L^2_xL^2_T}\big\|D_x^{s/2-1/4}u\big\|_{L_x^{4/(2s-1)}L_T^2}\\
&\quad +\ C\big\|D_x^{s/4-1/8}u\big\|_{L_x^{8/(5-2s)}L_T^2}\big\|D_x^{s/4-1/8}u\big\|_{L_x^{8/(6s-3)}L_T^2}\ \Big)\\
&\le CT^{1/2}\big\|D_x^{s-1/2}u\big\|_{L_T^\infty L_x^2}\Big(\,2\|u\|_{L^2_TL^2_x}\big\|D_x^{1/2}u\big\|_{L^2_TL^2_x}+C\|D_x^{s/2-1/4}u\big\|_{L^2_TL^2_x}\|D_x^{3/4-s/2}u\big\|_{L^2_TL^2_x}\ \Big)\\
&\le CT^{3/2}\|u\|^2_{L_T^\infty H_x^{s-1/2}}\|u\|_{L_T^\infty L_x^2}\|u\|_{L_T^\infty H_x^s}.
\end{aligned}
$$

From $1/2 < s \le 1$ we have $2 \le 4/(2s-1)$, $2 \le 8/(5-2s)$ and $2 \le 8/(6s-3)$. We have used this fact to apply Minkowski's and Sobolev's inequalities. Next by Hölder's and Minkowski's inequalities it follows that

$$
\begin{aligned}
A_{22} &\leq \big\|D_x^{s-1/2}u\big\|_{L_x^2L_T^2}\||u|^2\|_{L_x^2L_T^1}\\
&\leq CT^{1/2}\|u\|_{L_T^\infty H_x^{s-1/2}}\|u\|_{L_T^{4/3}L_x^2}\|u\|_{L_T^4L_x^\infty}\\
&\leq CT^{5/4}\|u\|_{L_T^\infty H_x^{s-1/2}}\|u\|_{L_T^\infty L_x^2}\|u\|_{L_T^4L_x^\infty}.
\end{aligned}
$$

Finally using Hölder's and Minkowski's inequalities and Proposition 3.4-(3.11) we have that

$$
\begin{aligned}
A_{23} &\leq \|u\|_{L_x^2L_T^2}\big\|D_x^{s-1/2}(|u|^2)\big\|_{L_x^2L_T^1}\\
&\leq T^{1/2}\|u\|_{L_T^\infty L_x^2}\big\|D_x^{s-1/2}(|u|^2)\big\|_{L_T^1L_x^2}\\
&\leq CT^{1/2}\|u\|_{L_T^\infty L_x^2}\|u\|_{L_T^4L_x^\infty}\big\|D_x^{s-1/2}u\big\|_{L_T^{4/3}L_x^2}\\
&\leq CT^{5/4}\|u\|_{L_T^\infty L_x^2}\|u\|_{L_T^\infty H_x^{s-1/2}}\|u\|_{L_T^4L_x^\infty}.
\end{aligned}
$$

Collecting the estimates for A_{21}, A_{22} and A_{23} we obtain the desired estimate (3.13). □

4. Local and Global Theory in $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$

In this section we consider the IVP (1.1) with data $u_0, v_0 \in L^2(\mathbb{R}^n),\ n = 1, 2, 3$. Our purpose is to prove Theorem 2.1. To do so we define an integral operator and a convenient space where this integral operator turns out to be a contraction operator. Using the contraction mapping principle we obtain the desired result.

We begin by defining the operator

$$
\Phi(u)(t) = S(t)u_0 - i\int_0^t S(t-t')\big(F_0(u(t')) + F_1(u(t'))\big)dt'. \tag{4.1}
$$

4.1. **Proof of Theorem 2.1.** For $R > 0$ and $T > 0$ we consider the function space

$$
E_T^R = \left\{u \in C([0,T] : L^2(\mathbb{R}^n)) \;/\; \|\!|u|\!\|_T \leq R\right\}, \tag{4.2}
$$

where

$$
\|\!|u|\!\|_T \equiv \|u\|_{L_T^\infty L_x^2} + \|u\|_{L_T^{8/n}L_x^4}. \tag{4.3}
$$

It is not difficult to show that E_T^R is a complete metric space.

It will be established that for appropriate choices of R and T, depending only on $\tau, \|u_0\|_{L^2}$ and $\|v_0\|_{L^2}$, that if $u \in E_T^R$ then $w = \Phi(u)$ belongs to E_T^R and $\Phi : E_T^R \to E_T^R$ is a contraction map. Thus most of what follows is the estimation of $\|\!|\Phi(u)|\!\|_T$. We need to bound the nonlinear term in (4.1).

First, by Proposition 3.1-3.2 and group properties we have that

$$
\|\!|S(t)u_0|\!\|_T \leq C_0\|u_0\|_{L^2}. \tag{4.4}
$$

Now we stimate the norm $\|\!|\cdot|\!\|_T$ of the inhomogeneous part in (4.1) corresponding to $F_0(u)$.

Proposition 3.1-(3.3) with $(r_1,q_1)=(\infty,2)$ and $(r_2,q_2)=(8/n,4)$ yields

$$\|\int_0^t S(t-t')F_0(u(t'))dt'\|_{L^\infty_T L^2_x} \leq C\|F_0(u)\|_{L_T^{8/(8-n)}L_x^{4/3}}. \tag{4.5}$$

Again by Proposition 3.1-(3.4), choosing $(r_1,q_1)=(r_2,q_2)=(8/n,4)$, it follows that

$$\|\int_0^t S(t-t')F_0(u(t'))dt'\|_{L_T^{8/n}L_x^4} \leq C\|F_0(u)\|_{L_T^{8/(8-n)}L_x^{4/3}}. \tag{4.6}$$

Using Hölder's and Minkowski's inequalities we obtain the following:

$$\begin{aligned}\|F_0(u)\|_{L_T^{8/(8-n)}L_x^{4/3}} &\leq \|e^{-t/\tau}v_0\|_{L_T^{4/(4-n)}L_x^2}\|u\|_{L_T^{8/n}L_x^4}\\ &\leq T^{(4-n)/4}\|v_0\|_{L^2}|\!|\!|u|\!|\!|_T.\end{aligned} \tag{4.7}$$

Hence (4.5), (4.6) and (4.7) yield

$$|\!|\!|\int_0^t S(t-t')F_0(u(t'))dt'|\!|\!|_T \leq C_1T^{(4-n)/4}\|v_0\|_{L^2}|\!|\!|u|\!|\!|_T. \tag{4.8}$$

Following the same arguments used to obtain (4.5) and (4.6) we get

$$\|\int_0^t S(t-t')F_1(u(t'))dt'\|_{L^\infty_T L^2_x} + \|\int_0^t S(t-t')F_1(u(t'))dt'\|_{L_T^{8/n}L_x^4} \leq C_2\|F_1(u)\|_{L_T^{8/(8-n)}L_x^{4/3}}. \tag{4.9}$$

On the other hand, Hölder's and Minkowski's inequalities give

$$\begin{aligned}\|F_1(u)\|_{L_T^{8/(8-n)}L_x^{4/3}} &\leq \tfrac{1}{\tau}\|u\|_{L_T^{8/n}L_x^4}\|\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'\|_{L_T^{4/(4-n)}L_x^2}\\ &\leq \tfrac{1}{\tau}\|u\|_{L_T^{8/n}L_x^4}T^{(4-n)/4}\|\int_0^t e^{-(t-t')/\tau}|u(t')|^2dt'\|_{L_T^\infty L_x^2}\\ &\leq \tfrac{1}{\tau}\|u\|_{L_T^{8/n}L_x^4}T^{(4-n)/4}\sup_{t\in[0,T]}\int_0^t e^{-(t-t')/\tau}\|u(t')\|^2_{L_x^4}dt'\\ &\leq \tfrac{1}{\tau}T^{(4-n)/4}\|u\|_{L_T^{8/n}L_x^4}T^{(4-n)/4}\|u\|^2_{L_T^{8/n}L_x^4}\\ &\leq \tfrac{1}{\tau}T^{(4-n)/2}|\!|\!|u|\!|\!|_T^3.\end{aligned} \tag{4.10}$$

The last estimate combined with estimate (4.9) implies

$$|\!|\!|\int_0^t S(t-t')F_1(u(t'))dt'|\!|\!|_T \leq \tfrac{C_2}{\tau}T^{(4-n)/2}|\!|\!|u|\!|\!|_T^3. \tag{4.11}$$

Hence (4.4), (4.8) and (4.11) yield

$$\begin{aligned}|\!|\!|\Phi(u)|\!|\!|_T &\leq C_0\|u_0\|_{L^2} + C_1T^{(4-n)/4}\|v_0\|_{L^2}|\!|\!|u|\!|\!|_T + \tfrac{C_2}{\tau}T^{(4-n)/2}|\!|\!|u|\!|\!|_T^3\\ &\leq C_0\|u_0\|_{L^2} + C_1T^{(4-n)/4}\|v_0\|_{L^2}R + \tfrac{C_2}{\tau}T^{(4-n)/2}R^3.\end{aligned} \tag{4.12}$$

Now we let $R = 2C_0\|u_0\|_{L^2}$. Therefore fixing T such that

$$C_1T^{(4-n)/4}\|v_0\|_{L^2} + \tfrac{C_2}{\tau}T^{(4-n)/2}R^2 \leq \frac{1}{2} \tag{4.13}$$

we conclude that $\Phi : E_T^R \to E_T^R$.

Using that

$$
\begin{aligned}
F_1(u) - F_1(\tilde{u}) =& \tfrac{\epsilon}{\tau}(u-\tilde{u}) \int_0^t e^{-(t-t')/\tau} |u|^2 dt' \\
&+ \tfrac{\epsilon}{\tau}\tilde{u} \int_0^t e^{-(t-t')/\tau} \left(\, \overline{u}(u-\tilde{u}) + \tilde{u}\left(\overline{u} - \overline{\tilde{u}}\,\right) \right) dt',
\end{aligned} \tag{4.14}
$$

a similar argument shows that

$$
\begin{aligned}
\|\Phi(u) - \Phi(\tilde{u})\|_T \le & \, C_1 T^{(4-n)/4} \|v_0\|_{L^2} \|u-\tilde{u}\|_T \\
&+ \tfrac{C_2}{\tau} T^{(4-n)/2} (\|u\|_T^2 + \|u\|_T \|\tilde{u}\|_T + \|\tilde{u}\|_T^2) \|u-\tilde{u}\|_T.
\end{aligned} \tag{4.15}
$$

Consequently $\Phi : E_T^R \to E_T^R$ is a contraction map and hence there exists a unique $u \in E_T^R$ with $\Phi(u) = u$.

A standard argument allow us to extend the result to the class

$$
E_T \equiv C([0,T] : L^2(\mathbb{R}^n)) \cap L^{8/n}([0,T] : L^2(\mathbb{R}^n)). \tag{4.16}
$$

Now we notice that by an analogous argument as used to estimate $\|\Phi(u)\|_T$ and using the fact that the unique solution u in E_T satisfies $u = \Phi(u)$, one can show that this solution also satisfies

$$
\|u\|_{L_T^r L_x^q} \le C_0 \|u_0\|_{L^2} + C_1 T^{(4-n)/4} \|v_0\|_{L^2} \|u\|_T + \tfrac{C_2}{\tau} T^{(4-n)/2} \|u\|_T^3 < \infty, \tag{4.17}
$$

where (r,q) is an admissible pair.

Next we show the persistence property of the solution $v(t)$ in L^2. Using (1.2), Minkowski's and Hölder's inequalities we have, for $t \in [0,T]$, that

$$
\begin{aligned}
\|v(t)\|_{L_x^2} &\le \|v_0\|_{L^2} + \tfrac{1}{\tau} \int_0^t e^{-(t-t')/\tau} \| |u|^2 \|_{L_x^2} \, dt' \\
&\le \|v_0\|_{L^2} + \frac{e^{-t/\tau}}{\tau} \Big(\int_0^t e^{\frac{4t'}{\tau(4-n)}} dt' \Big)^{(4-n)/4} \| |u|^2 \|_{L_T^{4/n} L_x^2} \\
&\le \|v_0\|_{L^2} + \left(\tfrac{4-n}{4}\right)^{\frac{4-n}{4}} \tfrac{1}{\tau^{n/4}} \|u\|^2_{L_T^{8/n} L_x^4}.
\end{aligned} \tag{4.18}
$$

Finally we show how to extend the solution to any time $T > 0$. We first note L^2-norm of the solution $u(t)$ is conserved which allows us to extended it to any positive time T. On the other hand, (4.18) guarantee that the solution $v(t)$ exists globally in L^2 and the proof of the Theorem 2.1 is completed.

5. Local and Global Theory in the One Dimensional Case

In this section we will prove Theorem 2.3 and 2.4 regarding local wellposedness for the IVP (1.3) in fractional Sobolev spaces. We use similar arguments as those given in the proof of

Theorem 2.1; but here the process is more delicate since we need to handle fractional derivatives. We also establish the global result advertised in Corollary 2.1.

5.1. **Proof of Theorem 2.3.** Let $s \in (0,1)$ and $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$. For $T > 0$ and $R > 0$ we define

$$X_T^R = \{u \in C([0,T] : H^s(\mathbb{R})) \;/\; \|\!|u|\!\|_T \le R\}, \tag{5.1}$$

where

$$\|\!|u|\!\|_T \equiv \|u\|_{L_T^\infty H_x^s} + \|u\|_{L_T^4 L_x^\infty}, \tag{5.2}$$

and we consider the map

$$\Phi(u)(t) = S(t)u_0 - i\int_0^t S(t-t')\big(F_0(u(t')) + F_1(u(t'))\big)dt'. \tag{5.3}$$

Now by Proposition 3.1-(3.2) and group properties we have that

$$\|\!|S(t)u_0|\!\|_T \le C_0\|u_0\|_{H^s}. \tag{5.4}$$

Proposition 3.1-(3.4) with $(r_2, q_2) = (4, \infty)$, Hölder's and Minkowski's inequalities yield

$$\begin{aligned}\|\int_0^t S(t-t')F_0(u(t'))\,dt'\|_{L_T^4 L_x^\infty} &\le C\,T^{1/4}\|e^{-t/\tau}v_0 u\|_{L_x^1 L_T^2}\\ &\le CT^{1/4}\|e^{-t/\tau}v_0\|_{L_x^2 L_T^\infty}\|u\|_{L_x^2 L_T^2}\\ &\le CT^{3/4}\|v_0\|_{L^2}\|\!|u|\!\|_T.\end{aligned} \tag{5.5}$$

Similarly, we obtain

$$\begin{aligned}\|\int_0^t S(t-t')F_1(u(t'))\,dt'\|_{L_T^4 L_x^\infty} &\le \tfrac{C}{\tau}T^{1/4}\|F_1(u)\|_{L_x^1 L_T^2}\\ &\le \tfrac{C}{\tau}T^{1/4}T^{5/4}\|u\|^2_{L_T^\infty L_x^2}\|u\|_{L_T^4 L_x^\infty}\\ &\le \tfrac{C}{\tau}T^{3/2}\|\!|u|\!\|_T^3,\end{aligned} \tag{5.6}$$

where we have used the proof of the Proposition 3.5-(ii) when $s = 1/2$.

Next, we estimate the nonlinear terms involving fractional derivatives. First we estimate the term corresponding to $F_0(u)$ for the inhomogeneous part of the equation (5.3). For this purpose we use that

$$\begin{aligned}&\|D_x^s\int_0^t S(t-t')F_0(u(t'))dt'\|_{L_x^2} = \|\int_0^t S(t-t')D_x^s F_0(u(t'))dt'\|_{L_x^2}\\ &\le \|\int_0^t S(t-t')\left\{D_x^s(e^{-t/\tau}v_0u) - e^{-t/\tau}v_0D_x^s u - uD_x^s(e^{-t/\tau}v_0)\right\}dt'\|_{L_x^2}\\ &+ \|\int_0^t S(t-t')(e^{-t/\tau}v_0D_x^s u)\|_{L_x^2} + \|\int_0^t S(t-t')(uD_x^s(e^{-t/\tau}v_0))\|_{L_x^2}\\ &\equiv B_1 + B_2 + B_3.\end{aligned} \tag{5.7}$$

For the first term we apply Minkowski's inequality, group properties, Proposition 3.4-(3.11), and Hölder's inequality in the time variable to obtain

$$\begin{aligned}
B_1 &\leq \|D_x^s(e^{-t/\tau}v_0u) - e^{-t/\tau}v_0D_x^su - uD_x^s(e^{-t/\tau}v_0)\|_{L_T^1L_x^2} \\
&\leq C\|u\|_{L_T^4L_x^\infty}\|D_x^s(e^{-t/\tau}v_0)\|_{L_T^{4/3}L_x^2} \\
&\leq CT^{3/4}\|u\|_{L_T^4L_x^\infty}\|D_x^sv_0\|_{L^2} \\
&\leq CT^{3/4}\|v_0\|_{H^s}\|\!|u|\!\|_T.
\end{aligned} \tag{5.8}$$

Using Proposition 3.1-(3.3) with $(r_1,q_1)=(\infty,2)$ and $(r_2,q_2)=(4,\infty)$ and Hölder's inequality we have

$$\begin{aligned}
B_2 &\leq C\|e^{-t/\tau}v_0D_x^su\|_{L_T^{4/3}L_x^1} \\
&\leq CT^{3/4}\|v_0\|_{L^2}\|D_x^su\|_{L_T^\infty L_x^2} \\
&\leq CT^{3/4}\|v_0\|_{L^2}\|u\|_{L_T^\infty H_x^s} \\
&\leq CT^{3/4}\|v_0\|_{L^2}\|\!|u|\!\|_T.
\end{aligned} \tag{5.9}$$

A similar argument shows that

$$\begin{aligned}
B_3 &\leq CT^{3/4}\|v_0\|_{H^s}\|u\|_{L_T^\infty L_x^2} \\
&\leq CT^{3/4}\|v_0\|_{H^s}\|\!|u|\!\|_T.
\end{aligned} \tag{5.10}$$

Gathering the information in (5.7)-(5.10) we get

$$\|D_x^s\int_0^t S(t-t')F_0(u(t'))dt'\|_{L_T^\infty L_x^2} \leq CT^{3/4}\|v_0\|_{H^s}\|\!|u|\!\|_T. \tag{5.11}$$

On the other hand, for the term corresponding to $F_1(u)$ we use Minkowski's inequality, group properties and Proposition 3.5-(i) to get

$$\begin{aligned}
\|D_x^s\int_0^t S(t-t')F_1(u(t'))dt'\|_{L_x^2} &\leq \|D_x^sF_1(u)\|_{L_T^1L_x^2} \\
&\leq \tfrac{C}{\tau}T^{3/2}\|u\|_{L_T^4L_x^\infty}^2\|u\|_{L_T^\infty H_x^s} \\
&\leq \tfrac{C}{\tau}T^{3/2}\|\!|u|\!\|_T^3.
\end{aligned} \tag{5.12}$$

Inserting the estimates (5.4)-(5.6) and (5.11)-(5.12) in the integral equation (5.3) it follows that

$$\begin{aligned}
\|\!|\Phi(u)|\!\|_T &\leq C_0\|u_0\|_{H^s} + C_1T^{3/4}\|v_0\|_{H^s}\|\!|u|\!\|_T + \tfrac{C_2}{\tau}T^{3/2}\|\!|u|\!\|_T^3 \\
&\leq C_0\|u_0\|_{H^s} + C_1T^{3/4}\|v_0\|_{H^s}R + \tfrac{C_2}{\tau}T^{3/2}R^3.
\end{aligned} \tag{5.13}$$

Now we let $R=2C_0\|u_0\|_{H^s}$. Therefore fixing T such that

$$C_1T^{3/4}\|v_0\|_{H^s} + \tfrac{C_2}{\tau}T^{3/2}R^2 \leq \frac{1}{2} \tag{5.14}$$

we conclude that $\Phi: X_T^R \to X_T^R$.

Using similar estimates we show that

$$\begin{aligned}\|\!|\Phi(u)-\Phi(\tilde{u})\|\!|_T &\le C_1 T^{3/4}\|v_0\|_{H^s}\|\!|u-\tilde{u}\|\!|_T \\ &+ \tfrac{C_2}{\tau}T^{3/2}(\|\!|u\|\!|_T^2+\|\!|u\|\!|_T\|\!|\tilde{u}\|\!|_T+\|\!|\tilde{u}\|\!|_T^2)\|\!|u-\tilde{u}\|\!|_T.\end{aligned} \tag{5.15}$$

Consequently $\Phi : X_T^R \to X_T^R$ is a contraction map and hence there exists a unique $u \in X_T^R$ with $\Phi(u) = u$.

Similarly as was estimated $\|\Phi(u)\|_{L_T^4 L_x^\infty}$ and using that the solution satisfies $\Phi(u) = u$ we obtain the additional regularity in (2.10).

Now we see that the solution $v(t)$ remains in H^s. Indeed, using (1.2), Hölder's inequality and Proposition 3.4-(3.11), for any $t \in [0,T]$ we have

$$\begin{aligned}&\|v(t)\|_{L^2}+\|D_x^s v(t)\|_{L^2} \\ &\le \|v_0\|_{L^2}+\|D_x^s v_0\|_{L^2}+\tfrac{1}{\tau}\int_0^t e^{-(t-t')/\tau}\||u|^2\|_{L^2}\,dt'+\tfrac{1}{\tau}\int_0^t e^{-(t-t')/\tau}\|D_x^s(|u|^2)\|_{L^2}\,dt' \\ &\le \|v_0\|_{H^s}+\tfrac{1}{\tau}\||u|^2\|_{L_T^1L_x^2}+\tfrac{1}{\tau}\|D_x^s(|u|^2)\|_{L_T^1L_x^2} \\ &\le \|v_0\|_{H^s}+\tfrac{1}{\tau}\|u\|_{L_T^4L_x^\infty}\|u\|_{L_T^{4/3}L_x^2}+\tfrac{C}{\tau}\|u\|_{L_T^4L_x^\infty}\|D_x^s u\|_{L_T^{4/3}L_x^2} \\ &\le \|v_0\|_{H^s}+\tfrac{C}{\tau}T^{3/4}\|u\|_{L_T^4L_x^\infty}\left(\|u\|_{L_T^\infty L_x^2}+\|D_x^s u\|_{L_T^\infty L_x^2}\right).\end{aligned} \tag{5.16}$$

Then the proof of Theorem 2.3 is completed.

Next prove local and global results when $s \in [1/2, 1]$ and $k \in I_s$. Here we use the same notation as in the proof of Theorem 2.3.

5.2. **Proof of Theorem 2.4.** First we let $\delta := \min\{1/4, k-s+1/2\}$ and we apply Proposition 3.3 with $\theta = 1-2\delta$, Proposition 3.4-(3.10) and Minkowski's, Hölder's and Sobolev's inequalities to obtain

$$\begin{aligned}&\|D_x^s\int_0^t S(t-t')F_0(u(t'))dt'\|_{L_x^2} = \|D_x^{1/2-\delta}\int_0^t S(t-t')(D_x^{s-1/2+\delta}(e^{-t/\tau}v_0u))dt'\|_{L_x^2} \\ &\le CT^\delta\|D_x^{s-1/2+\delta}(e^{-t/\tau}v_0u)\|_{L_x^{1/(1-\delta)}L_T^2} \\ &\le CT^\delta\Big(\|D_x^{s-1/2+\delta}u\|_{L_x^{2/(1-2\delta)}L_T^2}\|e^{-t/\tau}v_0\|_{L_x^2L_T^\infty} \\ &\qquad +\|u\|_{L_x^{2/(1-2\delta)}L_T^2}\|D_x^{s-1/2+\delta}(e^{-t/\tau}v_0)\|_{L_x^2L_T^\infty}\Big) \\ &\le CT^{\delta+1/2}\|v_0\|_{H^{s-1/2+\delta}}\|u\|_{L_T^\infty H_x^s} \\ &\le CT^{\delta+1/2}\|v_0\|_{H^k}\|\!|u\|\!|_T.\end{aligned} \tag{5.17}$$

Further by Proposition 3.3 and Proposition 3.5-(ii) we have

$$\begin{aligned}\|D_x^s\int_0^t S(t-t')F_1(u(t'))dt'\|_{L_x^2} &= \|D_x^{1/2}\int_0^t S(t-t')D_x^{s-1/2}F_1(u(t'))dt'\|_{L_x^2}\\ &\le \tfrac{C}{\tau}\|D_x^{s-1/2}F_1(u)\|_{L_x^1L_T^2}\\ &\le \tfrac{C}{\tau}(\,T^{5/4}+T^{3/2}\,)\|u\|^2_{L_T^\infty H_x^{s-1/2}}|\!|\!|u|\!|\!|_T.\end{aligned} \tag{5.18}$$

Therefore we see from (5.4)-(5.6) and (5.17)-(5.18) that

$$\begin{aligned}&|\!|\!|\Phi(u)|\!|\!|_T\\ &\le C_0\|u_0\|_{H^s}+C_1(T^{3/4}+T^{\delta+1/2}\,)\|v_0\|_{H^k}|\!|\!|u|\!|\!|_T+\tfrac{C_2}{\tau}(\,T^{5/4}+T^{3/2}\,)\|u\|^2_{L_T^\infty H_x^{s-1/2}}|\!|\!|u|\!|\!|_T\\ &\le C_0\|u_0\|_{H^s}+C_1(T^{3/4}+T^{\delta+1/2}\,)\|v_0\|_{H^k}R+\tfrac{C_2}{\tau}(T^{3/2}+T^{5/4}\,)R^3.\end{aligned} \tag{5.19}$$

Thus we first choose $R=2C_0\|u_0\|_{H^s}$ and then T satisfying

$$C_1(T^{3/4}+T^{\delta+1/2}\,)\|v_0\|_{H^k}+\tfrac{C_2}{\tau}(T^{3/2}+T^{5/4}\,)R^2\le\tfrac12. \tag{5.20}$$

Then we conclude that $\Phi: X_T^R\to X_T^R$. Similarly as in the previous cases we show that $\Phi: X_T^R\to X_T^R$ is a contraction map and hence there exists a unique $u\in X_T^R$ with $\Phi(u)=u$.

Now we let (r,q) with $q\in[2,\infty]$ and $2/r=1/2-1/q$. Using (3.5), (3.7), Proposition 3.1-(3.2, 3.4) and that the solution u satisfies $u=\Phi(u)$, we have

$$\begin{aligned}\|\partial_x u\|_{L_x^\infty L_T^2}+\|u\|_{L_T^rL_x^q} &\le C_0\|u_0\|_{H^{s-1/2}}+C_1\left(1+T^{1/4}\right)\|F_0(u)\|_{L_x^1L_T^2}\\ &\quad+\tfrac{C_2}{\tau}\left(1+T^{1/4}\right)\|F_1(u)\|_{L_x^1L_T^2}.\end{aligned} \tag{5.21}$$

Hence, the additional regularities in (2.14) hold.

Finally, using (1.2), Hölder's inequality, and Proposition 3.4-(3.11), for any $t\in[0,T]$ we have

$$\begin{aligned}&\|v(t)\|_{L^2}+\|D_x^k v(t)\|_{L^2}\\ &\le \|v_0\|_{L^2}+\|D_x^k v_0\|_{L^2}+\tfrac1\tau\int_0^t e^{-(t-t')/\tau}\||u|^2\|_{L^2}dt'+\tfrac1\tau\int_0^t e^{-(t-t')/\tau}\|D_x^k(|u|^2)\|_{L^2}dt'\\ &\le \|v_0\|_{H^k}+\tfrac1\tau\||u|^2\|_{L_T^1L_x^2}+\tfrac1\tau\|D_x^k(|u|^2)\|_{L_T^1L_x^2}\\ &\le \|v_0\|_{H^k}+\tfrac1\tau\|u\|_{L_T^4L_x^\infty}\|u\|_{L_T^{4/3}L_x^2}+\tfrac{C}{\tau}\|u\|_{L_T^4L_x^\infty}\|D_x^k u\|_{L_T^{4/3}L_x^2}\\ &\le \|v_0\|_{H^k}+\tfrac{C}{\tau}T^{3/4}\|u\|_{L_T^4L_x^\infty}\|u\|_{L_T^\infty H_x^s},\end{aligned} \tag{5.22}$$

where we also have used that $k\le s$ ($k\in I_s$). Hence $v(t)\in H^k$ in $[0,T]$, and we complete the proof of Theorem 2.3.

Finally we show how to extend the above solutions to any positive time T.

5.3. **Proof of Corollary 2.1.** Let $v_0 \in H^k$, $k \in I_s$ and let $[0,T^*)$ be the maximal time interval on which the Cauchy problem (1.3) has a unique solution $u \in X_T = C([0,T] : H^s(\mathbb{R})) \bigcap L^4([0,T] : L^\infty(\mathbb{R}))$ for any $T < T^*$. Suppose that $T^* < \infty$, we will show that it leads to contradiction.

First we note that the solution $u(x,t)$ of the IVP (1.3) satisfies $\Phi(u) = u$ and from (5.19) we see that

$$\begin{aligned} \|\!|u|\!\|_T &\le C_0\|u_0\|_{H^s} + C_1(T^{3/4} + T^{\delta+1/2})\|v_0\|_{H^k}\|\!|u|\!\|_T \\ &+ \tfrac{C_2}{\tau}(T^{3/2} + T^{5/4})\|u\|^2_{L^\infty_T H^{s-1/2}_x}\|\!|u|\!\|_T. \end{aligned} \tag{5.23}$$

for any $T < T^*$.

Now we consider two cases:

(i) $s = 1/2$ and $k \in [0,1/2]$.

From (5.23), using the conservation law (2.1) in L^2 and (5.23) we have that

$$\|\!|u|\!\|_T \le C_0\|u_0\|_{H^{1/2}} + \mu(T)\|\!|u|\!\|_T \tag{5.24}$$

where

$$\mu(T) = C_1(T^{3/4} + T^{1/2})\|v_0\|_{H^k} + \tfrac{C_2}{\tau}(T^{3/2} + T^{5/4})\|u_0\|^2_{L^2}. \tag{5.25}$$

Hence we can take $\tilde{T} \in [0,T^*]$ so that $\mu(\tilde{T}) \le 1/2$ with $\tilde{T}$ depending only on $\|u_0\|_{L^2}$ and $\|v_0\|_{H^k}$. Then from (5.24) we obtain

$$\|\!|u|\!\|_{T'} \le 2C_0\|u_0\|_{H^{1/2}} \tag{5.26}$$

for any $T' \in [0,\tilde{T}]$.

If $\tilde{T} = T^*$, we have that the solution u of the IVP (1.3) can be extended to the time interval $[0,T^*]$ with

$$\sup_{t\in[0,T^*]}\|u(t)\|_{H^{1/2}} \le 2C_0\|u_0\|_{H^{1/2}},$$

and we see that it contradicts the maximality of T^*. Therefore, suppose that $\tilde{T} < T^*$. Let $m \in \mathbb{N}$ be such that $T^* \le m\tilde{T}$ and replace $\tilde{T}$ by $\tilde{T} = T^*/m$.

Now consider the Cauchy problem

$$\begin{cases} i\partial_t u^{(1)} + \frac{1}{2}\partial_x^2 u^{(1)} = e^{-t/\tau}u^{(1)}v_0(x) + \frac{\epsilon}{\tau}u\int_0^t e^{-(t-t')/\tau}|u^{(1)}|^2 dt', \\ u^{(1)}(x,\tilde{T}) = u(x,\tilde{T}), \quad x \in \mathbb{R}. \end{cases} \tag{5.27}$$

Uniqueness of solutions yields that

$$\begin{cases} u(x,t), & t \in [0,\tilde{T}], \\ u^{(1)}(x,t), & t \in [\tilde{T}, 2\tilde{T}], \end{cases} \tag{5.28}$$

is a solution of IVP (1.3) in $[0,2\tilde{T}]$.

Using that $\|u_0\|_{L^2} = \|u(\tilde{T})\|_{L^2}$ and the same procedure to obtain (5.26), we have

$$\begin{aligned} \|\!|u|\!\|_{2\tilde{T}} &\le \max\left\{2C_0\|u_0\|_{H^{1/2}},\ 2C_0\|u(\tilde{T})\|_{H^{1/2}}\right\} \\ &\le \max\left\{2C_0\|u_0\|_{H^{1/2}},\ 4C_0^2\|u_0\|_{H^{1/2}}\right\}. \end{aligned} \tag{5.29}$$

Then, repeating this process m times, we see that

$$\|u\|_{T^*} \leq \max\left\{2C_0\|u_0\|_{H^{1/2}},\ 4C_0^2\|u_0\|_{H^{1/2}}, ..., \ (2C_0)^m\|u_0\|_{H^{1/2}}\right\} \tag{5.30}$$

which contradicts the maximality of T^*. Hence $T^* = +\infty$.

Finally, we observe that for any $T > 0$ we have

$$\|u\|_T \leq Q(T) := \max\left\{2C_0\|u_0\|_{H^{1/2}},\ 4C_0^2\|u_0\|_{H^{1/2}}, ..., \ (2C_0)^{m(T)}\|u_0\|_{H^{1/2}}\right\}$$

where $m(T) = \left[\frac{T}{\tilde{T}}\right] + 1$. Hence, without loss of generality, we may assume that $2C_0 > 1$ and consequently

$$\|u\|_{L^\infty_T H^{1/2}} \leq Q(T) \leq (2C_0)^{T/\tilde{T}+1}\|u_0\|_{H^{1/2}},$$

which gives (2.16) in Corollary (2.1).

(ii) $1/2 < s \leq 1$ and $k \in I_s = (s - 1/2, s]$.

Since $H^s \hookrightarrow H^{1/2}$, we may regard the solution as being contained in $H^{1/2}$. Moreover

$$\|u\|_{L^\infty_T H^{s-1/2}} \leq \|u\|_{L^\infty_T H^{1/2}} \leq Q(T). \tag{5.31}$$

Again, we suppose that $T_s^* < \infty$, where $[0, T_s^*)$ is the maximal time interval of existence of the solution.

Now we put

$$Q_0 := \sup\ \{Q(T) : T \in [0, T_s^*]\ \}. \tag{5.32}$$

Then from (5.23) and (5.31) for any $T \in [0, T_s^*)$ we have

$$\|u\|_T \leq C_0\|u_0\|_{H^s} + \mu(T)\|u\|_T \tag{5.33}$$

where

$$\mu(T) = C_1(T^{3/4} + T^{1/2}\ )\|v_0\|_{H^k} + \tfrac{C_2}{\tau}(T^{3/2} + T^{5/4}\ )Q_0^2. \tag{5.34}$$

Now using (5.33), (5.34), we can choose $\tilde{T}$, depending only on $\|v_0\|_{H^k}$ and Q_0, sufficiently small such that $\mu(\tilde{T}) \leq 1/2$, and consequently

$$\|u\|_{T'} \leq 2C_0\|u_0\|_{H^s} \tag{5.35}$$

for any $T' \in [0, \tilde{T}]$. Similar to case (**i**) we get $T_s^* = +\infty$.

Finally, we note that from (5.22) we have that the solution $v(t)$ exists globally in H^k. Then the proof of Theorem 2.4 is completed.

6. Local and Global Theory in $H^1(\mathbb{R}^n) \times H^1((\mathbb{R})^n$

Here we prove the Theorem 2.2. The proof of this theorem is similar to the proof of Theorem 2.1 so we only give a sketch of its proof.

6.1. **Proof of Theorem 2.2.** In this case, we define a new norm as follows:

$$\|u\|_T \equiv \|u\|_{L_T^\infty H_x^1} + \|u\|_{L_T^{8/n} L_x^4} + \|\nabla u\|_{L_T^{8/n} L_x^4}, \tag{6.1}$$

and for $R > 0$ and $T > 0$ we consider the function space

$$Y_T^R = \left\{ u \in C([0,T] : H^1(\mathbb{R}^n)) \;/\; \|u\|_T \le R \right\}. \tag{6.2}$$

First, by Proposition 3.1-(3.2) and group properties we have that

$$\|S(t)u_0\|_T \le C_0 \|u_0\|_{H^1}. \tag{6.3}$$

Since the norm $\|\Phi(u)\|_{L_T^\infty L_x^2} + \|\Phi(u)\|_{L_T^{8/n} L_x^4}$ was estimated in the proof of Theorem 2.1, we only need estimate the terms involving gradients.

Similarly as was obtained the estimates (4.8) and (4.11) we obtain

$$\begin{aligned}\|\nabla \int_0^t S(t-t')F_0(u(t'))dt'\|_{L_T^{8/n} L_x^4} &\le C\|\nabla F_0(u)\|_{L_T^{8/(8-n)} L_x^{4/3}} \\ &\le CT^{(4-n)/4}\|v_0\|_{H^1}(\|u\|_{L_T^{8/n} L_x^4} + \|\nabla u\|_{L_T^{8/n} L_x^4}) \\ &\le CT^{(4-n)/4}\|v_0\|_{H^1}\|u\|_T,\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\|\nabla \int_0^t S(t-t')F_1(u(t'))dt'\|_{L_T^{8/n} L_x^4} &\le C\|\nabla F_1(u)\|_{L_T^{8/(8-n)} L_x^{4/3}} \\ &\le \tfrac{C}{\tau} T^{(4-n)/2}\|u\|^2_{L_T^{8/n} L_x^4}\|\nabla u\|_{L_T^{8/n} L_x^4} \\ &\le \tfrac{C}{\tau} T^{(4-n)/2}\|u\|^2_{L_T^{8/n} L_x^4}\|u\|_T.\end{aligned} \tag{6.5}$$

Hence, from (6.3)-(6.5) we have

$$\|\Phi(u)\|_T \le C_0\|u_0\|_{H^1} + C_1 T^{(4-n)/4}\|v_0\|_{H^1}\|u\|_T + \tfrac{C_2}{\tau} T^{(4-n)/2}\|u\|^2_{L_T^{8/n} L_x^4}\|u\|_T, \tag{6.6}$$

and the rest of the proof of our local result is standard.

Now we extends the above solution to any positive time T.

We suppose the maximal time interval of existence of the solution, $[0, T_1^*]$, is finite. Using that our solution satisfies the integral equation $\Phi(u) = u$ and estimate (6.6), we have

$$\|u\|_T \le C_0\|u_0\|_{H^1} + \mu(T)\|u\|_T \quad \text{for} \quad T \in [0, T_1^*], \tag{6.7}$$

where

$$\mu(T) = C_1 T^{(4-n)/4}\|v_0\|_{H^1} + \tfrac{C_2}{\tau} T^{(4-n)/2}\|u\|^2_{L_T^{8/n} L_x^4}. \tag{6.8}$$

But we may regard the solution as being contained in L^2, then using Theorem 2.1 we see that the norm $\|u\|_{L_T^{8/n} L_x^4}$ is controlled globally by a constant. Similarly to the proof of Corollary 2.1, this fact yield a contradiction with $T_1^* < \infty$. Then the proof of Theorem 2.2 is completed.

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