EIGENVALUE ESTIMATES FOR HYPERSURFACES IN $\mathbb{H}^m \times \mathbb{R}$ AND APPLICATIONS

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Abstract. In this paper, we give a lower bound for the spectrum of the Laplacian on minimal hypersurfaces immersed into $\mathbb{H}^m \times \mathbb{R}$. As an application, in dimension 2, we prove that a complete minimal surface with finite total extrinsic curvature has finite index. On the other hand, for stable, minimal surfaces in $\mathbb{H}^3$ or in $\mathbb{H}^2 \times \mathbb{R}$, we give an upper bound on the infimum of the spectrum of the Laplacian and on the volume growth.

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1. Introduction

In this paper we give a lower bound on the infimum of the spectrum of the Laplacian Δ_g on a complete, orientable hypersurface (M^m, g) minimally immersed into $(\mathbb{H}^m \times \mathbb{R}, \hat{g})$ equiped with the product metric, with an application to the finiteness of the index in dimension 2. In dimension 2, under the assumption that the minimal surface is stable, we give an upper bound on the infimum of the spectrum and on the volume growth. We also consider the case when the minimal surface has finite index.

Let us fix some notations. Let ν denote a unit normal field along M and let $v = \hat{g}(\nu, \partial_t)$ be the component of ν with respect to the unit vector field ∂_t tangent to the $\mathbb{R}$-direction in the ambient space.

In Section 3, we give a lower bound of the spectrum of Δ_g which relies on the inequality $-\Delta_g b \geq (m-2) + v^2$ satisfied by a "horizontal" Busemann function b (see Proposition 3.1 and Corollary 3.2). In Section 4, we give two applications to minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. We prove that a complete minimal surface with finite total *extrinsic* curvature has finite index (Corollary 4.2) and we obtain a lower bound for the spectrum of the Laplacian on a complete minimal surface contained in a slab (Proposition 4.4).

In Section 5.1, we consider the operator $\Delta_g + a + bK_g$ on a complete Riemannian surface. When $a \geq 0$ and $b > 1/4$, we show that the positivity of this operator implies an upper bound on the infimum of the spectrum of Δ_g and on the volume growth of M (see Proposition 5.1 and Proposition 5.3). In Section 5.2, we apply these results to stable minimal surfaces in $\mathbb{H}^3$ or

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$\mathbb{H}^2 \times \mathbb{R}$, generalizing and extending results of A. Candel, [6]. Candel used Pogorelov's method, [19]. We use the method of Colding and Minicozzi, [10, 9].

In Section 6, we give some applications of our general lower bounds on the spectrum to higher dimensional hypersurfaces. In Section 2, we provide some preliminary technical lemmas.

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2. Preliminary computations

In this section, we make some preliminary computations for later reference. For the sake of simplicity, we work in the following model for the hyperbolic space $\mathbb{H}^{m+1}$,

$$(1) \quad \begin{cases} \mathbb{H}^{m+1} & = \mathbb{R}^m \times \mathbb{R}, \\ h & = e^{2s}(dx_1^2 + \cdots + dx_m^2) + ds^2 \text{ at the point } (x,s) \in \mathbb{H}^{m+1}. \end{cases}$$

These coordinates are known as "horocyclic coordinates" because the slices $\mathbb{R}^m \times \{s\}$ are horospheres and the coordinate function s is a Busemann function. They are quite natural when some Busemann function plays a special role, as will be the case in the sequel. Let γ_0 be the geodesic ray

$$(2) \quad \gamma_0 \ : \ \begin{cases} [0,\infty) \to \mathbb{H}^{m+1}, \\ u \mapsto \gamma_0(u) = (0,\ldots,0,u). \end{cases}$$

The Busemann function (see [1], p. 23) associated with γ_0 is the function

$$(3) \quad B \ : \ \begin{cases} \mathbb{H}^{m+1} \to \mathbb{R}, \\ (x,s) \mapsto B(x,s) = s. \end{cases}$$

In the sequel, we denote by

$$(4) \quad \begin{cases} D^h & \text{the Levi-Civita connexion,} \\ \Delta_h & \text{the geometric (\textit{i.e.} non-negative) Laplacian,} \end{cases}$$

for the hyperbolic metric h on $\mathbb{H}^{m+1}$.

Lemma 2.1. *With the above notations, we have the formulas,*

$$(5) \qquad \Delta_h B = -m,$$

$$(6) \qquad \mathrm{Hess}_h B = e^{2s}(dx_1^2 + \cdots + dx_m^2)$$

at the point $(x,s) \in \mathbb{H}^{m+1}$. In particular, if we decompose the vector $u \in T_{(x,s)}\mathbb{H}^{m+1}$ h-orthogonally as $u = (u_x, u_s)$, we have,

$$(7) \qquad \mathrm{Hess}_h B(u,u) = h(u_x, u_x).$$

The proof is straightforward. □

Recall the following general lemmas.

Lemma 2.2. *Let $(M^m, g) \looparrowright (\widehat{M}^{m+1}, \hat{g})$ be an orientable isometric immersion with unit normal field ν and corresponding normalized mean curvature H. Let $\widehat{F} : \widehat{M} \to \mathbb{R}$ be a smooth function and let $F := \widehat{F}|_M$ be its restriction to M. Then, on M,*

$$\Delta_g F = \Delta_{\hat{g}} \widehat{F}|M + \mathrm{Hess}_{\hat{g}} \widehat{F}(\nu, \nu) - mH d\widehat{F}(\nu).$$

Proof. See for example [11], Lemma 2. □

Lemma 2.3. *Assume that the manifold (M, g) carries a function f which satisfies*

$$|df|_g \leq 1 \quad \textit{and} \quad -\Delta_g f \geq c \quad \textit{for some constant} \quad c > 0.$$

Then, any smooth, relatively compact domain $\Omega \subset M$ satifies the isoperimetric inequalities

$$\mathrm{Vol}_{m-1}(\partial\Omega) \geq c \, \mathrm{Vol}_m(\Omega) \quad \textit{and} \quad \lambda_1(\Omega) \geq \frac{c^2}{4},$$

where $\lambda_1(\Omega)$ is the least eigenvalue of Δ_g in Ω, with Dirichlet boundary condition.

Proof. Integration by parts and Cauchy-Schwarz. □

3. Hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$

We consider orientable, isometric immersions $(M^m, g) \looparrowright (\widehat{M}^{m+1}, \hat{g})$, with unit normal ν, where $\widehat{M} = \mathbb{H}^m \times \mathbb{R}$ with the product metric $\hat{g} = h + dt^2$. We take the model (1) for the hyperbolic space (here with dimension m), so that $\widehat{M}$ is the product $\mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}$, with the Riemannian metric $\hat{g}$ given by

$$\hat{g} = e^{2s}(dx_1^2 + \cdots + dx_{m-1}^2) + ds^2 + dt^2.$$

We define the function $\hat{b}$ on $\widehat{M}$ by

$$\hat{b}(x_1, \ldots, x_{m-1}, s, t) = s. \tag{8}$$

This function is in fact a Busemann function of $\widehat{M}$ (seen as a Cartan-Hadamard manifold) associated with a "horizontal" geodesic (justifying the name "horizontal" Busemann function used in the introduction).

We call $b := \hat{b}|_M$ the restriction of $\hat{b}$ to M. We decompose the unit vector ν according to the product structure $\mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}$, orthogonally with respect to $\hat{g}$, as

$$\nu = \nu_x + w\partial_s + v\partial_t. \tag{9}$$

Applying Lemma 2.2, we obtain the equation

$$\Delta_g b = \Delta_{\hat{g}} \hat{b}|_M + \mathrm{Hess}_{\hat{g}} \hat{b}(\nu, \nu) - mH\hat{g}(\nu, \partial_s). \tag{10}$$

Using (7) and (9), it can we rewritten as

$$-\Delta_g b = (m-1) - |\nu_x|^2 + mHw, \tag{11}$$

and we note that $|\nu_x|^2 + v^2 + w^2 = 1$. It follows that

$$-\Delta_g b \geq (m-2) + v^2 + w^2 - mH|w|. \tag{12}$$

For minimal hypersurfaces, we deduce from (12) the following results.

Proposition 3.1. *Let $(M^m, g) \looparrowright (\mathbb{H}^m \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal hypersurface, with normal vector ν. Recall that $v = \hat{g}(\nu, \partial_t)$. Then,*

$$-\Delta_g b \geq (m-2) + v^2. \tag{13}$$

Corollary 3.2. *Let $(M^m, g) \looparrowright (\mathbb{H}^m \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal hypersurface, with normal vector ν. Let $v = \hat{g}(\nu, \partial_t)$. Let $\lambda_\sigma(\Delta_g)$ be the infimum of the spectrum of the Laplacian Δ_g on M. Then*

$$\lambda_\sigma(\Delta_g) \geq \Big(\frac{m-2+\inf_M v^2}{2}\Big)^2 \geq \Big(\frac{m-2}{2}\Big)^2. \tag{14}$$

Corollary 3.3. *Let $(M^m, g) \looparrowright (\mathbb{H}^m \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal hypersurface, with $m \geq 3$. Then (M, g) is non-parabolic.*

Proof. Apply Proposition 10.1 of [15] using (14). □

When the mean curvature H is non-zero, we also obtain the following result from inequality (12),

Proposition 3.4. *Let $(M^m, g) \looparrowright (\mathbb{H}^m \times \mathbb{R}, \hat{g})$ be a complete, orientable hypersurface, with normal vector ν and constant mean curvature H, $0 \leq H \leq \frac{m-1}{m}$. Recall that $v = \hat{g}(\nu, \partial_t)$. Then,*

$$-\Delta_g b \geq (m-2)(1-\sqrt{1-v^2}) + (m-2)\Big(1 - \frac{mH}{m-2}\Big)\sqrt{1-v^2}. \tag{15}$$

Remarks. (i) Inequalities (13) and (14) are sharp. Indeed, take the horizontal slice $M = \mathbb{H}^m \times \{0\}$, in that case $v = 1$, or take $M = \mathbb{P} \times \mathbb{R}$, where $\mathbb{P}$ is some totally geodesic $(m-1)$-space in $\mathbb{H}^m$, in that case $v = 0$. (ii) In dimension 2, Corollary 3.2 is empty in general. However, inequality (13) is useful even in dimension 2, as we will show in Section 4. (iii) Inequality (15) generalizes an earlier result of the second author ([8]) for submanifolds immersed in Hadamard manifolds. For other estimates, see also [5]. We point out that it is more convenient in our context to use the "horizontal" Busemann function rather than the hyperbolic distance function as in [8]. (iv) The above inequalities still hold if M^m is only assumed to have mean curvature bounded from above by H.

4. Applications to minimal hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$

4.1. **Index of minimal surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$.** The stability operator of a minimal hypersurface $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$ is given by

$$J_M = \Delta + (m-1)(1-v^2) - |A|^2, \tag{16}$$

where v is the vertical component of the unit normal ν, and A the second fundamental form of the immersion (see [3]). It turns out that the spectrum of the operator $\Delta + (m-1)(1-v^2)$ is bounded from below by a positive constant. More precisely, we have the following result.

Proposition 4.1. *Let $(M^m, g) \looparrowright (\mathbb{H}^m \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal hypersurface with normal vector ν. Let $v = \hat{g}(\nu, \partial_t)$. Then the spectrum of the operator $\Delta_g + (m-1)(1-v^2)$ on M is bounded from below by $(\frac{m-1}{2})^2$.*

Proof. We start from the inequality (13), $-\Delta_g b \geq (m-2)+v^2$. We multiply this inequality by f^2, where $f \in C_0^\infty(M)$, and integrate by parts using the fact that $|db|_g \leq 1$. We obtain (all integrals are taken with respect to the Riemannian measure dv_g),

$$(m-2)\int_M f^2 + \int_M v^2 f^2 \leq \int_M |df^2| \leq 2\int_M |f||df|.$$

We re-write this inequality as

$$(m-1)\int_M f^2 \leq 2\int_M |f||df| + \int_M (1-v^2)f^2.$$

Using the Cauchy-Schwarz inequality $2|f|.|df| \leq \frac{1}{a}|df|^2 + af^2$ for $a > 0$, we obtain

$$a(m-1-a)\int_M f^2 \leq \int_M (|df|^2 + a(1-v^2)f^2) \leq \int_M (|df|^2 + (m-1)(1-v^2)f^2),$$

provided that $0 \leq a \leq m-1$. We can now maximize the constant in the left-hand side by choosing $a = (m-1)/2$. □

Remark. We observe that equality is achieved in the above inequality when M is a slice $\mathbb{H}^m \times \{t_0\}$, in which case $v = 1$. If we assume that $v^2 \leq \alpha^2 < 1$, the spectrum of $\Delta_g + (m-1)(1-v^2)$ is bounded from below by $(m-1)(1-\alpha^2)$.

Corollary 4.2. *Let $(M^2, g) \looparrowright (\mathbb{H}^2 \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal surface, with second fundamental form A. If $\int_M |A|^2 dv_g$ is finite, then the immersion has finite index.*

Proof. When $\int_M |A|^2$ is finite, the second fundamental form tends to zero uniformly at infinity (see [3], Theorem 4.1). Using Proposition 4.1 with $m = 2$, it follows that the essential spectrum of the Jacobi operator J_M is bounded from below by $\frac{1}{4}$. Since the operator J_M is also bounded from below, it follows that it has only finitely many negative eigenvalues (see [2], Proposition 1). □

Remark. This corollary answers a question raised in [3], where the finiteness of the index of J_M is proved in dimension $m \geq 3$ under the assumption that $\int_M |A|^m$ is finite, and in dimension 2 under the assumption that both $\int_M v^2$ and $\int_M |A|^2$ are finite. In dimension $m \geq 3$, the index of J_M is bounded from above by a constant times $\int_M |A|^m$ (see [3]). In the next section, we investigate bounds on the index in dimension 2.

4.2. Bounds on the index of minimal surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$.

Proposition 4.3. *Let $(M^2, g) \looparrowright (\mathbb{H}^2 \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal surface, with second fundamental form A. If $\int_M |A|^2\, dv_g$ is finite, then for any $r > 1$, there exists a constant C_r such that the index of the immersion is bounded from above by $C_r \int_M |A|^{2r}\, dv_g$.*

Remarks. (i) Recall that the assumption that $\int_M |A|^2\, dv_g$ is finite implies that A tends to zero uniformly at infinity. It follows that the integrals $\int_M |A|^{2r}\, dv_g$ are all finite. (ii) Our proof provides a constant C_r which tends to infinity when r tends to 1. We do not know whether there is a bound of the index in terms of $\int_M |A|^2 \, dv_g$ as this is the case for minimal surfaces in $\mathbb{R}^3$ (see [22]).

Proof. As in Section 4.1, we write the Jacobi operator as $J = \Delta_g + 1 - v^2 - |A|^2$. The closure $\widetilde{Q}$ of the quadratic form $Q[f] = \int_M \left(|df|^2 + (1-v^2)f^2\right) dv_g$ with domain $C_0^1(M)$ satisfies the Beurling-Deny condition (if f is in the domain of $\widetilde{Q}$, then so is $|f|$ and $\widetilde{Q}[|f|] = \widetilde{Q}[f]$, see [13], Theorem 1.3.2) and, by Proposition 4.1, the Cheeger inequality

$$\int_M f^2 \, dv_g \leq 4Q[f], \quad \forall f \in C_0^1(M). \tag{17}$$

On the other-hand, the surface M satisfies the Sobolev inequality

$$\int_M f^2 \, dv_g \leq S\big(\int_M |df|_g^2 \, dv_g\big)^2, \quad \forall f \in C_0^1(M), \tag{18}$$

for some constant $S > 0$. Indeed, this follows from the Sobolev inequality for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, using the fact that the ambient space has non-positive curvature and infinite injectivity radius (see [16]).

From the above Cheeger and Sobolev inequalities, we can establish that for any $q \geq 1$, there exists a constant D_q such that for any $f \in C_0^1(M)$,

$$\big(\int_M |f|^{2q} \, dv_g\big)^{1/q} \leq D_q Q[f]. \tag{19}$$

When q is an integer, the inequality follows from an induction argument and we can conclude by interpolation.

We can then apply Theorem 1.2 of [18] to conclude that the index is less than $e^p D_q^p \int_M |A|^{2p} \, dv_g$ where $p = q/(q-1)$. $\square$

4.3. **Hypersurfaces in a slab.** In this section, we use the computations of Section 3 to give a lower bound on the spectrum of the Laplacian on a complete minimal surface immersed in a slab $\mathbb{H}^2 \times [-a, a]$, $a > 0$.

Let us first consider functions on $\mathbb{H}^m \times \mathbb{R}$ depending only on the height t, namely $\hat{\beta}(x, s, t) = f(t)$. In this case, $d\hat{\beta} = f'(t)dt$, and

$$\mathrm{Hess}_{\hat{g}} \hat{\beta}(X, Y) = f''(t)\hat{g}(X, \partial_t)\hat{g}(Y, \partial_t).$$

In particular,

$$\Delta_{\hat{g}} \hat{\beta} = -f''(t) \quad \text{and} \quad \mathrm{Hess}_{\hat{g}} \hat{\beta}(\nu, \nu) = v^2 f''(t).$$

Let us define $\beta = \hat{\beta}|_M$. Using Lemma 2.2, we have

$$-\Delta_g \beta = (1 - v^2) f''(t) + mHvf'(t). \tag{20}$$

In order to estimate the first eigenvalue of a *minimal* hypersurface $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$, we use the identity (20) with some particular choice of f. For instance, let $\hat{\beta}(x, s, t) = \frac{1}{2}t^2$. In this case, we have

$$-\Delta_g \beta = (1 - v^2).$$

Assume now that $M^m \looparrowright \mathbb{H}^m \times [-a, a]$, for some $a > 0$. Then,

$$-\Delta_g \beta = (1 - v^2) \ \text{ and } \ |d\beta| \leq a.$$

If we define $Z = b + \beta$, where b is the restriction of the Busemann function $\hat{b}$ to M^m, we can use the last inequality in (12) to obtain

$$-\Delta Z \geq m - 1 \quad \text{and} \quad |dZ| \leq \sqrt{1 + a^2}. \tag{21}$$

Using the above notation and Lemma 2.3, we have the following estimate,

Proposition 4.4. *Given $a > 0$, let $(M^m, g) \looparrowright (\mathbb{H}^m \times [-a, a], \hat{g})$ be a complete, immersed, orientable, minimal hypersurface. Then, the infimum of the spectrum of Δ_g on M is positive. More precisely,*

$$\lambda_\sigma(\Delta_g) \geq \frac{(m-1)^2}{4(1 + a^2)}. \tag{22}$$

5. Bounds derived from a stability assumption

Let (M, g) be a complete Riemannian surface with (non-negative) Laplace operator Δ_g and Gaussian curvature K_g. Let a, b be real numbers, with $a \geq 0$ and $b > 1/4$. Let L be the operator $L = \Delta_g + a + bK_g$.

Let $\operatorname{Ind}(L, \Omega)$ denote the number of negative eigenvalues of the operator L in Ω, with Dirichlet boundary conditions on $\partial\Omega$. The index, $\operatorname{Ind}(L)$, of the operator L is defined to be the supremum

$$\operatorname{Ind}(L) = \sup\{\operatorname{Ind}(L, \Omega) \mid \Omega \Subset M\}$$

taken over the relatively compact subdomains Ω in M.

In Section 5.1, we state two intrinsic consequences of the assumption that the operator L has finite index. In Sections 5.2 and 5.3, we consider applications to minimal and CMC surfaces.

5.1. **Intrinsic results.**

Proposition 5.1. *Let (M, g) be a complete non-compact Riemannian surface. Let $a \geq 0$ and $b > \frac{1}{4}$. Denote by Δ_g the (non-negative) Laplacian and by K_g the Gaussian curvature of (M, g). Denote by $\lambda_\sigma(\Delta_g)$ the infimum of the spectrum of Δ_g and by $\lambda_e(\Delta_g)$ the infimum of the essential spectrum of Δ_g.*

(1) *If the operator $\Delta_g + a + bK_g$ is non-negative on $C_0^\infty(M)$, then,*

$$\lambda_\sigma(\Delta_g) \leq \frac{a}{4b - 1}.$$

(2) *If the operator $\Delta_g + a + bK_g$ has finite index on $C_0^\infty(M)$ and if M has infinite volume, then,*

$$\lambda_e(\Delta_g) \leq \frac{a}{4b - 1}.$$

Proof. The proof uses the method of Colding-Minicozzi [10], and more precisely Lemma 1.8 in the second author's paper [9].
Proof of Assertion 1. We can assume the surface to have infinite volume (otherwise $\lambda_\sigma(\Delta_g) = 0$ because the function 1 is in $L^2(M, v_g)$ and the estimate is trivial). Fix a point $x_0 \in M$ and let $r(x)$ denote the Riemannian

distance to the point x_0. Given $S > R > 0$, let $B(R)$ denote the open geodesic ball in M with center x_0 and radius R. Let $C(R,S)$ denote the open annulus $B(S) \setminus \bar{B}(R)$. Let $V(R)$ denote the volume of $B(R)$ and $L(R)$ the length of its boundary $\partial B(R)$. Let $G(R)$ denote the integral curvature of $B(R)$, $G(R) = \int_{B(R)} K_g(x)\, dv_g(x)$, where dv_g denotes the Riemannian measure. The main idea in [9] is to use the work of Shiohama-Tanaka [20, 21] on the length of geodesic circles, where it is shown that the function $L(r)$ is differentiable almost everywhere and related to the Euler characteristic and to the integral curvature of geodesic balls by the formula ([9], Theorem 1.7)

$$L'(r) \leq 2\pi\chi(B(r)) - G(r) \leq 2\pi - G(r),$$

where the second inequality comes from the fact that the Euler characteristic of balls is less than or equal to 1. Recall the following lemma.

Lemma 5.2 (Lemma 1.8. in [9])**.** *For $0 < R < S$, let $\xi : [R,S] \to \mathbb{R}$ be such that $\xi \geq 0, \xi' \leq 0$, $\xi'' \geq 0$ and $\xi(S) = 0$. Then*

$$\begin{aligned} \int_{C(R,S)} K_g \xi^2(r)\, dv_g \leq \quad & -\xi^2(R)G(R) + 2\pi\xi^2(R) - 2\xi(R)\xi'(R)L(R) \\ & - \int_{C(R,S)} (\xi^2)''(r)\, dv_g. \end{aligned}$$

To prove Assertion 1,we choose ξ as in Lemma 5.2, and a function $f : B(S) \to \mathbb{R}$ such that $f(r) \equiv \xi(R)$ on $B(R)$, $f(r) = \xi(r)$ on $C(R,S)$, and we write the positivity assumption,

$$0 \leq \int_M |df|_g^2\, dv_g + a \int_M f^2\, dv_g + b \int_M K_g f^2\, dv_g.$$

On the ball $B(R)$, we have

$$\int_{B(R)} K_g f^2\, dv_g = \xi^2(R)G(R) \quad \text{and} \quad \int_{B(R)} |df|^2\, dv_g = 0.$$

Using Lemma 5.2, we obtain

$$\begin{aligned} 0 \leq \quad & \int_{C(R,S)} (\xi')^2(r)\, dv_g + a \int_M f^2\, dv_g + b\xi^2(R)G(R) - b\xi^2(R)G(R) \\ & +2\pi b\xi^2(R) - 2b\xi(R)\xi'(R)L(R) - b\int_{C(R,S)} (\xi^2)''(r)\, dv_g, \end{aligned}$$

and hence,

$$\begin{aligned} 0 \leq \quad & (1-2b)\int_{C(R,S)} (\xi')^2(r)\, dv_g + a\int_M f^2\, dv_g \\ & +2\pi b\xi^2(R) - 2b\xi(R)\xi'(R)L(R) - 2b\int_{C(R,S)} \xi(r)\xi''(r)\, dv_g. \end{aligned} \tag{23}$$

We choose $\xi(r) = (S-r)^k$ in $[R,S]$ for $k \geq 1$ big enough (we will eventually let k tend to infinity). Then $\xi(r)\xi''(r) = (1 - \frac{1}{k})(\xi'(r))^2$. It follows that

$$\begin{aligned} 0 \leq \quad & (1 - 4b + 2b/k)\int_M |df|^2\, dv_g + a\int_M f^2\, dv_g \\ & +2b\Big(\pi(S-R)^{2k} + kL(R)(S-R)^{2k-1}\Big). \end{aligned}$$

Using the fact that $\int_M f^2\, dv_g \geq (S-R)^{2k}V(R)$, we obtain

$$\begin{aligned} \lambda_\sigma(\Delta_g) \quad & \leq \frac{\int_M |df|^2\, dv_g}{\int_M f^2\, dv_g} \\ & \leq \frac{a}{4b-1-2b/k} + \frac{2b}{(4b-1-2b/k)V(R)}(\pi + \tfrac{kL(R)}{S-R}). \end{aligned} \tag{24}$$

We first let S tend to infinity, then we let R tend to infinity, using the fact that M has infinite volume, and we let finally k tend to infinity to obtain

$$\lambda_\sigma(\Delta_g) \leq \frac{a}{4b-1}.$$

Proof of Assertion 2. It is a well-known fact that the finiteness of the index of the operator $\Delta_g + a + bK_g$ implies that it is non-negative outside a compact set (see [14], Proposition 1). We choose R_0 big enough for $\Delta_g + a + bK_g$ to be non-negative in $M \setminus B(R_0)$. Next, for $S > R > R_1 + 1 > R_0 + 1$, we choose ξ as in Lemma 5.2, and a test function f as follows

$$f(r) = \begin{cases} 0 & \text{in } B(R_1), \\ \xi(R)(r - R_1) & \text{in } C(R_1, R_1 + 1), \\ \xi(R) & \text{in } C(R_1 + 1, R), \\ \xi(r) & \text{in } C(R, S). \end{cases} \tag{25}$$

Following the same scheme as for Assertion 1, and under the assumption that the volume of M is infinite, we can prove that the bottom of the spectrum of Δ_g in $M \setminus B(R_1)$, with Dirichlet boundary conditions on $\partial B(R_1)$, satisfies the inequality

$$\lambda_\sigma(\Delta_g, M \setminus B(R_1)) \leq \frac{a}{4b-1}.$$

To conclude, we use the fact that

$$\lambda_e(\Delta_g) = \lim_{R \to \infty} \lambda_\sigma(\Delta_g, M \setminus B(R)).$$

□

Proposition 5.3. *Let (M, g) be a complete Riemannian surface with (non-negative) Laplace operator Δ_g and Gaussian curvature K_g. Let $V(r)$ denote the volume of the geodesic ball of radius r in M (with center some given point x_0). Let a, b be positive real numbers, with $b > 1/4$. Let $\alpha_0 = \sqrt{a/(4b-1)}$. If the operator $L := \Delta_g + a + bK_g$ has finite index, then*

$$\forall \alpha > \alpha_0, \qquad \int_0^\infty e^{-2\alpha r} V(r)\, dr < \infty,$$

and hence, the lower volume growth of M satisfies

$$\liminf_{r \to \infty} r^{-1} \ln(V(r))) \leq 2\alpha_0.$$

Proof. It follows from our assumptions that the operator L is positive outside some compact set (see [14], Proposition 1). In particular, it is positive on $M \setminus B(R_0)$ for some radius R_0. Choose $R > R_0 + 1$ and define the function

$$\xi(r) = \begin{cases} 0 & \text{for} \quad r \leq R_0, \\ (1 - \frac{R_0+1}{R})^{\alpha R}(r - R_0) & \text{for} \quad R_0 \leq r \leq R_0 + 1, \\ (1 - \frac{r}{R})^{\alpha R} & \text{for} \quad R_0 + 1 \leq r \leq R, \end{cases} \tag{26}$$

where the parameter α will be chosen later on. The positivity of the operator L on $M \setminus B(R_0)$ implies that

$$0 \leq \int_M \Big((\xi'(r))^2 + a\xi^2(r) + bK_g \xi^2(r) \Big)\, dv_g.$$

We write the integral on the right-hand side as the sum of two integrals, $\int_{C(R_0,R_0+1)}$ and $\int_{C(R_0+1,R)}$. The first integral can be written as

$$\int_{C(R_0,R_0+1)} = \Big(1-\frac{R_0+1}{R}\Big)^{\alpha R} C(B(R_0)),$$

where $C(B(R_0))$ is a constant which only depends on the geometry of M on the ball $B(R_0)$. Using Lemma 5.2 and the fact that $\chi(B(r)) \le 1$ for all r, the second integral can be estimated as follows

$$\begin{aligned}\int_{C(R_0+1,R)} \le\ & \int_{C(R_0+1,R)} \Big((\xi')^2 + a\xi^2 - b(\xi^2)''\Big)\, dv_g \\ & +2\pi b - \xi^2(R_0+1)G(R_0+1) + 2\alpha L(R_0+1).\end{aligned}$$

Using (26), the definition for the function ξ, the integral in the first line of the above inequality can be written as

$$-\Big((4b-1)\alpha^2 - \frac{2b\alpha}{R} - a\Big)\int_{R_0+1}^{R} (1-\frac{r}{R})^{2\alpha R-2} L(r)\, dr.$$

Taking α big enough so that the constant is positive, and using the fact that $L(r) = V'(r)$, we obtain the inequality

$$\frac{2\alpha R-2}{R}\Big((4b-1)\alpha^2 - \frac{2b\alpha}{R} - a\Big)\int_{R_0+1}^{R}\Big(1-\frac{r}{R}\Big)^{2\alpha R-3} V(r)\, dr \le D(B(R_0),\alpha),$$

where $D(B(R_0),\alpha)$ is a constant which only depends on the geometry of M in the ball $B(R_0)$ and α. Letting R tend to infinity, we finally obtain that

$$2\alpha\Big((4b-1)\alpha^2 - a\Big)\int_{R_0+1}^{\infty} e^{-2\alpha r} V(r)\, dr < \infty,$$

provided that $\alpha > \alpha_0$, which proves the first assertion in the theorem. The second assertion follows easily. ☐

Remark. In the above theorem, we have assumed that $a > 0$. In the case $a = 0$, one can show that the volume growth is at most quadratic (see [9], Proposition 2.2).

5.2. **Applications to stable minimal surfaces in $\mathbb{H}^3$ or $\mathbb{H}^2\times\mathbb{R}$.** Let M be a complete, orientable, minimal immersion into either the 3-dimensional hyperbolic space $\mathbb{H}^3$ or into $\mathbb{H}^2\times\mathbb{R}$. Let J_M denote the Jacobi operator of the immersion.

In the case of a minimal immersion $M \looparrowright \mathbb{H}^3(-1)$, the operator J_M takes the form $J_M = \Delta_M + 2 - |A|^2$, where A is the second fundamental form. Using the Gauss equation, we have that $K_M = -1 - \frac{1}{2}|A|^2$, so that we can rewrite the Jacobi operator of $M \looparrowright \mathbb{H}^3(-1)$ as

$$J_M = \Delta_M + 4 + 2K_M. \tag{27}$$

In the case of a minimal immersion $M \looparrowright \mathbb{H}^2(-1)\times\mathbb{R}$, the Jacobi operator is given by $J_M = \Delta_M + 1 - v^2 - |A|^2$, where v is the vertical component of the unit normal vector to the surface. Using the Gauss equation, we have that $K_M = -v^2 - \frac{1}{2}|A|^2$, so that we can rewrite the Jacobi operator of $M \looparrowright \mathbb{H}^2(-1) \times \mathbb{R}$ as

$$J_M = \Delta_M + 2 + 2K_M - (1 - v^2) \leq \tilde{J}_M := \Delta_M + 2 + 2K_M. \tag{28}$$

In this case, the positivity of the operator J_M implies the positivity of the operator $\tilde{J}_M$.

Applying Proposition 5.1 to the operator J_M in the form (27) when M is a minimal surface in $\mathbb{H}^3$, *resp.* to the operator $\tilde{J}_M$ in the form (28) when M is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$, we obtain the following proposition.

Proposition 5.4. *Let $(M, g) \looparrowright (\widehat{M}, \hat{g})$ be a complete, orientable, minimal immersion. Assume that the immersion is stable.*

(1) *If $\widehat{M} = \mathbb{H}^3$, then $\lambda_\sigma(\Delta_g) \leq \frac{4}{7}$.*
(2) *If $\widehat{M} = \mathbb{H}^2 \times \mathbb{R}$, then $\lambda_\sigma(\Delta_g) \leq \frac{2}{7}$.*

If the immersion is only assumed to have finite index, then the same inequalities hold with $\lambda_\sigma(\Delta_g)$ replaced by $\lambda_e(\Delta_g)$, the infimum of the essential spectrum.

Remarks. (i) The first assertion improves an earlier result of A. Candel [6] who proved that $\lambda_\sigma(M) \leq \frac{4}{3}$, provided that M is a complete, *simply-connected*, stable minimal surface in $\mathbb{H}^3$. (ii) Note that in both cases, the bottom of the spectrum of a totally geodesic $\mathbb{H}^2$ is 1/4.

Applying Proposition 5.3, we have the following proposition.

Proposition 5.5. *Let $(M, g) \looparrowright (\widehat{M}, \hat{g})$ be a complete, orientable, minimal immersion. Let μ denote the lower volume growth rate of M,*

$$\mu = \liminf_{r \to \infty} r^{-1} \ln(V(r))),$$

where $V(r)$ is the volume of the geodesic ball $B(x_0, r)$ for some given point x_0. Assume that the immersion has finite index.

(1) *If $\widehat{M} = \mathbb{H}^3$, then $\mu \leq 2\sqrt{\frac{4}{7}}$.*
(2) *If $\widehat{M} = \mathbb{H}^2 \times \mathbb{R}$, then $\mu \leq 2\sqrt{\frac{2}{7}}$.*

Remarks. (i) Assertion 1 in Proposition 5.5 improves a previous result in [6], where Candel gives an upper bound on μ under the assumption that M is simply-connected. (ii) Recall from [17, 4] that the volume growth is related to the infimum of the essential spectrum by the formula

$$\lambda_e(\Delta_g) \leq \Big(\frac{\liminf_{r \to \infty} r^{-1} \ln(V(r)))}{2}\Big)^2.$$

5.3. **Futher applications.** We note that the above argument also works for surfaces with constant mean curvature $|H| \leq 1$ in hyperbolic space. In that case, $K_M = -(1 - H^2) - \frac{1}{2}|A|^2$ and $J_M = \Delta_M + 4(1 - H^2) + 2K_M$. So that, we obtain the following proposition.

Proposition 5.6. *Let $(M,g) \looparrowright \mathbb{H}^3$ be a complete, orientable, stable CMC immersion, with $|H| \leq 1$. Then*

$$\lambda_\sigma(\Delta_g) \leq \frac{4(1-H^2)}{7}.$$

The space $\mathbb{H}^2 \times \mathbb{R}$ is a simply-connected 3-dimensional homogeneous manifold, whose isometry group has dimension 4. Such manifolds have been well studied (see for instance [12] and references therein) and can be parametrized by two real parameters, say κ and τ, with $\kappa \neq 4\tau^2$. We denote them by $\mathbb{E}^3(\kappa,\tau)$. When $\tau = 0$, $\mathbb{E}^3(\kappa, 0) =$ is the product space $\mathbb{E}^2(\kappa) \times \mathbb{R}$, where $\mathbb{E}^2(\kappa)$ is the space form of constant curvature κ. In particular, $\mathbb{H}^2 \times \mathbb{R} = \mathbb{E}^3(-1, 0)$.

If $(M,g) \looparrowright \mathbb{E}^3(\kappa,\tau)$ is an immersed CMC H surface, then its Jacobi operator is given by (see [12], Proposition 5.11)

$$J_M := \Delta_g + 2K - 4H^2 - \kappa - (\kappa - 4\tau^2)v^2.$$

In the next proposition we give an upper bound for the bottom of the spectrum in this general framework.

Proposition 5.7. *Let $(M,g) \looparrowright \mathbb{E}^3(\kappa,\tau)$ be a complete, orientable, stable CMC H immersion, such that $\kappa < 4\tau^2$. Assume furthermore that $2H^2 \leq (2\tau^2 - \kappa)$. Then*

$$\lambda_\sigma(\Delta_g) \leq \frac{4\tau^2 - 2\kappa - 4H^2}{7}.$$

Proof. Under the hypotheses we have the follows inequalities:

$$0 \leq \Delta_g + 2K - 4H^2 - \kappa - (\kappa - 4\tau^2)v^2 \leq \Delta_g + 2K - 4H^2 - 2(\kappa - 2\tau^2),$$

and we may apply Proposition 5.1 again. □

6. Applications in higher dimensions

In this Section, we give some further applications of the inequalities we proved in Section 3. In the following proposition, we give a structure theorem for minimal hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$.

Proposition 6.1. *Let $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$, $m \geq 3$, be a complete, orientable minimal hypersurface, with unit normal field ν and second fundamental form A. Let v denote the component of ν along ∂_t. For $0 \leq \alpha \leq 1$, there exists a constant $c(m,\alpha)$ satisfying $c(m,\alpha) > 0$, whenever*

(1) *$m \geq 7$ and $\alpha \geq 0$,*
(2) *$m = 6$ and $\alpha \geq 0.083$,*
(3) *$m = 5$ and $\alpha \geq 0.578$.*

If M satisfies $\|A\|_m \leq c(m,\alpha)$ and $v^2 \geq \alpha^2$, then M carries no L^2-harmonic 1-form and hence has at most one end.

Proof. We only sketch the proof. The proof uses several ingredients.

1. According to [16], the manifold M^m satisfies the Sobolev inequality

$$\|\varphi\|^2_{\frac{2m}{m-2}} \leq S(2,m)\|d\varphi\|_2^2, \quad \forall \varphi \in C^1_0(M). \tag{29}$$

2. Let $u \in T_1M$ be a unit tangent vector to M. By Gauss equation, we have the relation

$$\mathrm{Ric}(u,u) = \widehat{\mathrm{Ric}}(u,u) - \widehat{R}(u,\nu,u,\nu) - |A(u)|^2,$$

where Ric denotes the Ricci curvature of M, $\widehat{\mathrm{Ric}}$ the Ricci curvature and $\widehat{R}$ the curvature tensor of $\widehat{M} = \mathbb{H}^m \times \mathbb{R}$, and where A denotes the Weingarten operator of the immersion. Using the curvature computations in [3] and the fact that A has trace zero, we obtain the inequality

$$\mathrm{Ric}(u,u) \geq -(m-1) - \frac{m-1}{m}|A|^2. \tag{30}$$

Let ω be an L^2 harmonic 1-form on M. Using the Weitzenböck formula for 1-forms, the improved Kato inequality

$$\frac{1}{m-1}|d|\omega||^2 \leq |D\omega|^2 - |d|\omega||^2, \tag{31}$$

and inequality (30), we find that ω satisfies the following inequality in the weak sense

$$\frac{1}{m-1}|d|\omega||^2 + |\omega|\Delta|\omega| \leq (m-1)|\omega|^2 + \frac{m-1}{m}|A|^2|\omega|^2. \tag{32}$$

The following formal calculation can easily be made rigorous by using cut-off functions. Integrate (32) over M using integration by parts and the notation $f := |\omega|$,

$$\frac{m}{m-1}\int_M |df|^2 \leq (m-1)\int_M f^2 + \frac{m-1}{m}\int_M |A|^2 f^2.$$

Plug the assumption $|v| \geq \alpha$ and the inequality (14) into the preceding inequality. Use Hölder's inequality to estimate the integral $\int_M |A|^2 f^2$ and the Sobolev inequality (29). It follows that

$$[\frac{m}{m-1} - \frac{4(m-1)}{(m-2+\alpha)^2}]\|f\|^2_{\frac{2m}{m-2}} \leq S(2,m)\frac{m-1}{m}\|A\|^2_m\|f\|^2_{\frac{2m}{m-2}}$$

and we can conclude the proof with the constant

$$C(m,\alpha) = \frac{m}{(m-1)(S(2,m))}\,\frac{m(m-2+\alpha)^2 - 4(m-1)^2}{(m-1)(m-2+\alpha)^2}.$$

□

Proposition 6.2. *Let $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$ be a complete, orientable minimal hypersurface, with second fundamental form A. Assume that $||A||_m < \infty$. Then*

1. *M^m has finite index, if $m \geq 3$,*
2. *M^m has only finitely many ends, if $m \geq 7$.*

Proof. Assertion 1 was proved in [3]. To prove Assertion 2, we can mimic the proof of Corollary 4.2 to show that the operator $L := \Delta + \frac{\sqrt{m-1}}{2}|A|^2 - (m-1)$ has finite index, when $m \geq 7$. We then apply Theorem 1 of [7] to conclude the proof. □

Proposition 6.3. *Let $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$, $m \geq 3$, be a complete, orientable minimal hypersurface, with unit normal field ν and second fundamental form A. Let v denote the component of ν along ∂_t. If,*

(1) $\|A\|_\infty \leq (\frac{m-1}{2})^2$, *or*
(2) $\|A\|_\infty \leq (\frac{m-2+\alpha}{2})^2$ *and* $v^2 \geq \alpha$, *or*
(3) $|A|^2 + (m-1)v^2 \leq \frac{m^2}{4}$ *on* M.

then the immersion M is stable.

Proof. Recall that the Jacobi operator J_M of the immersion M is given by the formula

$$J_M = \Delta_g + (m-1)(1-v^2) - |A|^2.$$

Assertion 1 follows from Proposition 4.1. Assertions 2 and 3 follow from Corollary 3.2. □

Remark 1. The second condition is not so interesting because it implies that v does not vanish. If M is connected, we may assume that $v > 0$ and then M is stable because v is a Jacobi field, $J_M(v) = 0$.

Remark 2. We can write the operator J_M as

$$J_M = \Delta_g - (\frac{m-2}{2})^2 + [(\frac{m}{2})^2 - |A|^2].$$

In view of the results à la Lieb or Li-Yau, one can show that if the integral

$$\int_M [(\frac{m}{2})^2 - |A|^2]_-^{m/2}$$

is small enough, then M is stable.

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